

SOCF5110

DfcZYgg]cbU`DfUWfJWY'F YgYUfVX'DfcYWh

HYfa "Hk c" ## "&\$%-

7ci fgY⁻Cj Yfj]Yk

GHJZZ:7cbHJWh:8YHJ]`g`

'7cbj Ybcfg'

B Ua Y	9a U]`	5j U]`UV]`]hm	@cWJhjcb	D\cbY
AUifYYb'AUW]b`Ym	a Ui fYYb"a UW[]b`Ym4 i bgk "YXi "	K YXbYgXUmgʻ-Uaʻ!	A cfj Yb 6fck b	\$&
	Ui	%da 'cf'Vm	; %(- ' ,) , ,
		Uddc]bha Ybh		

GWXcc`7cbHJWhi±bZcfa Uh]cb

GWXcc``cZGcWJU``GWJYbWYg

Fcca '%) -

A cfj Yb 6fck b 7 &\$

Ya U]`. 'gcgg4 i bgk "YXi "Ui

d\cbY. '\$&'-',) '% \$+

7ci fgY 8YHJ] g

7 fYX]hDc]bhg * *

Gi a a UfmcZh\Y 7ci fgY

GcWJU~K cf_

H\]g`]g`U''&`I C 7 DfcZYgg]cbU``DfUW]WY`F YgYUfW\ DfcYWiWci fgY"'Mci ik]``Ybfc``]b`U'*'i cWWci fgY']b YUW\ cZhk c`gi WWgg]j Y`hYfa g`k]h\`U'fYgi `hfYdcfhYX`VmU'g]b[`Y`[fUXY`Uhih\Y'YbX`cZh\Y'gYWcbX`*'i cW Wci fgY"'\do h\]g`Wci fgYz`GC 7:) %\$`DfcZYgg]cbU``DfUW]WY`F YgYUfW\ 'DfcYWz`mci ik]```Wca a YbW'`UbX`k cf_ hck UfXg`Ub']bXYdYbXYbhfYgYUfW\ 'dfcYWik]h\`U'hcd]WX]fYWfmfY`UhYX`hc`mci f`dfcZYgg]cbU``dfUW]WY _bck `YX[Y`]b`7 ci bgY``]b['GcW]U``K cf_z`7 ci d`Y`UbX`: Ua]`mH\YfUdmz`cf`GcW]U``8 Yj Y`cda Ybh'`Hcd]W Uddfcj U``]g`fYei]fYX`Vmh\Y`fY`Yj UbhA UghYfg`7 ci fgYk cf_'Dfc[fUa '7 ccfX]bUhcf"'DfcYWfYdcfhk cfX `Yb[h\`]g`%&z\$\$\$!%(z\$\$\$`k cfXgz`UbX`j Uf]Uh]cb`k]h\]b`h\]g`fUb[Y`]g`hc`VY`bY[ch]UhYX`k]h\`h\Y`Gi dYfj]gcf UWWcfX]b['hc`h\Y`gdYWZ]WfYei]fYa Ybhg`cZh\Y`fYgYUfW\ 'dfcYWfUbX`]hg`a Yh\cXc`c[m'

7 ci fgY @YUfb]b[Ci hWca Yg

- %" 8 Yg][bž'XYj Y`cdž'UbX'fYdcfhicb'U'fYgYUfW\\'dfc'YWhfY`UhYX'hc'gc\\J\\U`k cf_'UbX\cf'\Wci bgY``]b[h\Ycfmcf'dfU\\J\\\"
- &" FYj]Yk 'h\Y``]hYfUhi fY'cb'U'fYgYUfW\'hcd]WfY`UhYX'hc'gcWJU``k cf_'UbX#cf'Wci bgY``]b['h\Ycfmicf dfUWJWY"
- ' " 8Ya cbglfUhY`_bck`YX[Y`UbX`i bXYfglUbX]b[`cZU`WcfY`UfYU`cZgcW]U``k cf_`UbX#cf`Wci bgY``]b[h\YcfmUbX`dfUW]WY"

HYUWX]b['GhfUhY[]Yg'

 $\label{thm:linear} H\Y'ZYYXVUW_g\YYhg\#i\ Vf]Wg'k\]```VY'Uj\ U]`UV`Y'hc'ghi\ XYbhg'Uhih\Y'ghUfhicZh\Y'Wci fgY'gc'h\Uhih\Ym'WUb'k cf_hck UfXg'gdYW]ZYX'ghUbXUfXg''$

5XX]h]cbU`XYhU]`g.

H\Y`fYdcfhff\Yg]gŁk]``fYei]fY`ghiXYbhg`hc`Wcad`YhY`Ub`]bhfcXiWf]cb`UbX`fUh]cbU`Y`Zcf`h\Y]f`dfcdcgYXfYgYUfW.`hcd]Wfl\z\$\$\$`—'&z\$\$\$`k cfXgŁ/U`WcadfY\Ybg]j Y`fYj]Yk `cZh\Y`fY`Yj Ubh`]hYfUhi fY`ff*z\$\$\$`—', z\$\$\$ k cfXgŁ/UbXž`U`dfcdcgYX`aYh\cXc`c[mZcf`ZhifYfYgYUfW.`h\Uh]bWiXYg`U`Zi```Wcbg]XYfUh]cb`cZYh\]WU`Wcbg]XYfUh]cbg`f\z\$\$\$`k cfXgŁ"`Gi[[YghYX`k cfX``Yb[h\]g]bX]WUh]j Y`cb`m"

 $H\Y'dfc\Y'WiWkcgYb`UbX'hkY'hkYg]g`g\ci`X`gYfjY`Ug`U'dfYdUfUhjcb`ghYd`hckUfX`U`8GK`cf`D\8`fYgYUfWkghiXm''H\Y`fYdcfhik]```Wcadf]gY`%$$i``cZh\Y`[fUXY'''Hkc'aUf_Yfg`k]```UggYgg`h\Y`fYdcfhiUbX`h\Y`Z]bU``aUf_k]```fYZYWfh\Y`X]ZZYfYbW'`VYfkYYb`h\Y`hkc'aUf_g`UkUfXYX'''H\Y`aUf_]b[``Wf]hYf]U'UfY`VY`ck.``$

7 f]hYf]cb	,) Ž	+)!,(*)	0*)।
C f][]bU`]hm	Hcd]WWYUf'm cf][]bU`k cf_ž dfcj]X]b['bYk]bg][\hg']bhc'h\Y gi VYWia UhYf"	Hcd]Wg\ck g Yj]XYbWY cZ cf][]bU`]lmžYl hYbXg _bck `YX[Y`]b h\Y Z]Y`X	G\ck gca Y XY[fYY cZcf][]bU`]hmVi h a U]b`mZc``ck g YI]gh]b[gh X]Yg"	@jthY cf bc Yj]XYbWY cZcf][]bU` WcbHYbH'
I gY'cZh\Ycfm# a Yh\cXc`c[m .	±bHY[fUHYg'h\Ycfm]bhc'h\Y'X]gW gg]cb dUdYf'k Y`"'G\ck g h\cfci[\ i bXYfgHJbX]b['cZ YI]gh]b['h\Ycfm" ±b'Ya d]f]WJ'h\YgYg h\Y'a Yh\cXc`c[m]g k Y``!W\cgYb'UbX Wta dYHYbhm dfYgYbHYX"	H\Ycfmei]hY k Y``]bhY [fUhYX]bhc h\Y X]gW gg]cb" G\ck g [ccX i bXYfghUbX]b[cZ h\Ycfm' D Ya d]f]WU h\YgYg h\Y a Yh\cXc`c[m dfcdcgYX]g gci bX"	H\YcfmdUfhm]bHY[fUHYX]bhc X]gW gg]cb"G\ck g cb`mVUg]W i bXYfgHJbX]b[cZ h\Ycfm' DYa d]f]WJ`h\YgYg h\Y a YH\cXc`c[m dfcdcgYX]g	

			VY 'WYUfYf"	
Gci fWg	≢bg][\hZi``UbX	Gci bX fYj]Yk cZ	5XYei UhY fYj]Yk cZ	Dccf ⁻ fYj]Yk ⁻ cZ
	h\cfci [\'fYj]Yk'cZ	YI]gh]b[``]hYfUhifY	YI]gh]b[``]hYfUhifY	Yl]gh]b[``]hYfUhifY
	`]hYfUhifY'cb'h\Y		Vi hg\ck]b[ZYk	k]h\ 'g][b]Z]\Wbh
	hcd]W]bg][\hg"	ca]gg]cbg"
7 cblf]Vi h]cb hc	Di V`]g\UV`Y`]b`U	Di V`]g\UV`Y`]b`U	I b`]_Y`mihc VY	Bchdi V`]g\UV`Y
_bck `YX[Y	ĉi fbU`k]h\	ĉi fbU`Vi hk]h\	di V`]g\UV`Y'Yj Yb	
	a]b]a U`	gca Y #a cXYfUhY	k]h\giVghUbh]U`	
	a cX]ZJWUhjcb UbX	a cX]ZJWUhjcb UbX	a cX]ZJWUhjcb	
	g\Ud]b["	g\Ud]b[

 $\label{eq:higher_problem} \mbox{Hi fb]h]b "gYhh]b[. "H\]g"]g"bch"U"Hi fb]h]b"Ugg][\mbox{ba Ybh}$

FYgci fWyg

DfYgWf]VYX F Ygci fWYg

BchUj U]`UV`Y

FYWca a YbXYX FYgci fWyg

FYWca a YbXYX FYgci fWfg.

%"6fma Ubž'5"f&\$%&L"GcWJU`fYgYUfWX a Yh\cXg`f(h\ YX"L"ClZcfXž'9b[`UbX. ClZcfX`lb]j Yfg]lm"

&"`GdfYb_`Yž'8"<"'/ `D]YfWhž': "D"`flYXgŁ'f&\$\$) Ł"`F YgYUfW\``A Yh\cXg`]b`: Ua]`mH\YfUdmž'; i]`ZcfX`DfYgg.

 $B Yk `Mcf_"`H\g'di V`]WUh]cb`WUb`VY'di fVX UgYX'cb!`]bY`Vm`UWWgg]b[`h\Y'Zc``ck]b[`I F@ \\ \hd.#k k k "eVX"Wca "Ui #dfcXi W# +, %) +&' $-*$- `F YgYUfWXSA Yh\cXgS+bS: Ua]`mSH\YfUdmSVmSGdf Yb_`YS8 ci [`UgS< $98 HSD]YfWhS: fYXSD$98 HS"\ha$

5XX]h]cbU`fYUX]b[g.

%"'6UVV]Yž'9"'f&\$\$+Ł"'H\Y'dfUWf)WY'cZgcW]U`fYgYUfWX'fW%th\'YX"Ł"'6Y`acbh'KUXgkcfh\"

&"`BYi a UbžK "@"f&\$\$' Ł"`GcWJU``fYgYUfW\`a Yh\cXg.`E i U`]hUh]j Y`UbX`ei Ubh]hUh]j Y`UddfcUW\Yg`ff'h\YX"Ł" 6cghcb.`5``mb'/ '6UWcb"

' "`FiV]bž'5"ž/ '6UVV]Yž'9"`F"'f&\$\$, Ł"`FYgYUfW\`aYh\cXg'Zcf'gcW]U``kcf_`ff'h\`YX"Ł"'6Y`acbhž'75.H\cagcb<|[\Yf'9XiWUh]cb"

("'<c`cg_cž'A"'>"'f&\$\$*\L''Df]a Yf'Zcf'Wf]h]ei]b['gcW]U``fYgYUfW..'5'ghi XYbh'[i]XY"'6Y`a cbhž'75.H\ca gcb

.

Gh XYbhgʻUfYʻYbWci fU[YXˈhcˈUWWYggʻch\Yfʻa UhYf]U`h\fci [\'UWUXYa]W^ci fbU`gʻUbXʻh\Y`]bhYfbYh' gYZ ``^ci fbU`gʻ]bWi XY.

GcWU`K cf_FYgYUfW

EiU`]hUh]jY' ⇒bei]fm

6f]h]g\'>cifbU\'cZGcW]U\'K cf_

>ci fbU`cZGcWJU`K cf_'9Xi WUhJcb

>cifbU`cZGcWJU`GYfj]WYFYgYUfWX

FYqYUfWX cb GcWJU K cf_DfUWJWY

GcWU`K cf

±bhYfbUhcbU``>cifbU``cZGcWJU``KY`ZJfY

9j U`i Uhjcb'FYj]Yk '>ci fbU``cZ5dd`]YX'FYgYUfW\` ⇒hYfbUhjcbU``>ci fbU``cZGcWjU``FYgYUfW\`A Yh\cXc`c[m: Ua]`mDfcWygg

6f]h]g\`>cifbU``cZ: Ua]`m'H\YfUdm

5i glfU`]Ub'UbX'BYk 'NYU`UbX'>ci fbU`cZ: Ua]`mH\YfUdm

D`YUgY'bchYz'U'F Yj]Yk 'cZZJa]`mz'Wci d`Yg'UbX'gmghYa]Wh\YfUdmci hWca Y'fYgYUfW\'&\$\$\$!&\$\$-'VmDYhYf GhfUhrcbz'9a a U'G]'j Yfz'BUhUg\U'BUgW]a Ybhcz'; k Yb'Dck Y``z'@]n'A W8cbbY``'/ '9k U'Bcj chbm]g'Uj U]`UV'Y'Uh\hhd.#k k k "UZh'cf["i #fU]b]b[#fYgYUfW\"Ugd

FYWca a YbXYX FYgci fWg Zcf gdYWZWUfYUg]bWi XY.

FYgYUfWX '9h\]Wq

5bhYž'6"`>"/ FY[Y\fž'7"'f&\$\$' Ł"'6YmcbX']bX]j]Xi U``f][\hg'UbX'ZfYYXca g. 'a YhU!Yh\]Wg']b'gcW]U``k cf_fYgYUfW\"'GcW]U``K cf_ž'(, ž'%)!%(("

6i hYfž="f&\$\$&\z6f]n]g\`>ci fbU``cZGcW]U``k cf_`UbX`gcW]U``WUfY`fYgYUfW\ž6f]n]g\`>ci fbU``cZGcW]U``K cf_" ' ž&' -!&(, "

F Ua W. UfUbžD"/ '7 i hW]ZYž>"F"'f&\$\$\\">i X[]b['h\Y'Yh\]Wg'cZei U`]hUhj Y'fYgYUfW. 'Wcbg]XYf]b['h\Y'Yh\]Wg'dfcW'gg'a cXY`"'<YU'h\/ 'GcW]U`'7 UfY']b'h\Y'7 ca a i b]hm-ž') \% **"

 $A \ Uf[\ c`]b\check{z}; \ "`7\]Yb\check{z}8"\check{z}8i \ a \ Ub\check{z}G": \ Ui \ W\&]Yf\check{z}'5"; \ cfX]g\check{z}'9"C`]j \ Yf\check{z}D"`F \ Ua \ cg\check{z}A "`UbX`J]W_Yfa \ Ub\check{z}'?" f\&\$\$) \&"``9h\&]WU``=ggi \ Yg`]b`7ci d`Y`UbX': \ Ua]`mF \ YgYUfW&'\check{z}'>ci \ fbU``cZ: \ Ua]`mDgmW\&c`c[m\check{z}Jc`"`% \check{z}'Bc"`% dd"`%) +-%* +$

9j]XYbWY!VUqYX`dfUWhJWY/ fYZYWhJj Y`fYqYUfWX

7 f]gdž'6"'f&\$\$(\text{L"'9j}]XYbWY!VUgYX'dfUWJWY'UbX'h\Y'VcfXYfg'cZXUhU']b'h\Y'[`cVU``]bZcfa UhJcb'YfU"'>ci fbU`cZGcWJU`K cf_'9Xi WUhJcbz'(\\$z'+'!, *"

 $: cc_\check{z} > "fPX" + fW--* + "HV'fYZYWN Yf. GcW]U`k cf_Yfg' + kYcf]Yg`cZdfUWN WfYgYUfWX" Gh @YcbUfXg. 5``Yb'/ T bk]b"$

;] $VVg\check{z}@''/'$; Ua Vf]`` $\check{z}'9''$ 'f&\$\$&&"'9j]XYbWY!VUgYX'dfUW]WY. `Wci bHYfUf[i a Ybhg'hc'cVYW]cbg''`F YgYUfWX'cbGcW\U``K cf 'DfUW\W\\check{z}'\&\z'() &!(+*"

A W8cbU`Xž7"'f&\$\$' E": cfk UfX'j]U'h\Y'dUgh8'9j]XYbW!VUgYX'dfUWfW'Ug'ghfUhY[m]b'gcW]U`k cf_"'H\Y 8fUk]b['6cUfX.'5b'5i ghfU']Ub'F Yj]Yk 'cZDi V`]W5ZZJ]fqž' ž'%k' !%(&"

D`Uh\ž'8"`f&\$\$*\L"'9j]XYbWY!VUgYX`dfUWI)WY.`W ffYbh]ggi Yg`UbX`Z h fY`X]fYWI)cbg"'5i ghfU`]Ub`GcW]U``K cf_ž)-\ž')*!+&"

FcgYbž'5"'f&\$\$' \text{2"'9j}XYbWY!VUgYX'gcWJU``kcf_'dfUWJWY.'W\U``Yb[Yg'UbX'dfca]gY"'GcWJU``Kcf_'FYgYUfW\ž\&+ž'%+!&\$,"

K YVVžG":5":f&\$\$%L":Gca Y:Wcbg]XYfUh]cbg:cb:h\Y:j U`]X]hmcZYj]XYbWY!VUgYX:dfUWf]WY:]b:gcW]U`:k cf_"

6f]h]g\`>cifbU``cZGcW]U``K cf_ž'' %ž')+!+-"

7 cfVmž·6"'f&\$\$*±":5dd`m]b['fYgYUfW\']b'gcW]U``k cf_'dfUW]W\"`A U]XYb\YUX. `C dYb'I b]j Yfg]m''; f]bbY``ž`F " A "'f9X"±'f%-, , ±"'GcW]U``k cf_'fYgYUfW\'UbX'Yj U`i Uh]cb"' ≢UgWU.': 9 'DYUWċW_"

A Ufick žī7 "F "f&\$\$) Ł"F YgYUfW. a Yh\cXg`Zcf`[YbYfU`]ghigcWJU``k cf_"'6Y`a cbh'6fcc_g#7 c`Y"'B Yi a Ubž'K " @'/ '? fYi [Yfž'@'`K "'f&\$\$' Ł"'GcWJU``k cf_'fYgYUfW. a Yh\cXg. 'E i U`]hUhjj Y`UbX'ei UbhjhUhjj Y`Udd`]WUhjcbg" 6cghcb. 'DYUfgcb'9Xi WUhjcb"

F cmgYž'8 "`f&\$\$(Ł"`F YgYUfWX 'a Yh\cXg`]b`gcWJU``k cf_`f(h\`YX"Ł"`DUWJZJW; fcj Y. '6fcc_g#7 c`Y"`GUfUbHJ_cgž G"`f&\$\$) Ł"`GcWJU``fYgYUfWX 'fl fX`YX"Ł" @cbXcb. 'A UWa]``Ub"

K U`hYfž'A "`fPX"Ł'f&\$\$*Ł"`GcWJU``fYgYUfW\`a Yh\cXg.`Ub`5i ghfU`]Ub`dYfgdYWfjj Y"`Gci h\`A Y`Vci fbY.`C I ZcfX

FYWca a YbXYX cb`]bY a UhYf]U`g UbX ch\Yfi gYZi `gci fWYg

- %" GcWYmrZcf'GcWJU`K cf_'UbX'F YgYUfWX'! \hd.#k k k "ggk f"cf[#
- &" 9j]XYbWY`6UqYX`DfUWf\WY`]b`< YU`h\`7UfY`!`\htd.#k k k "V]ca YX"`]V"i a b"YXi #YUfb#YVd#
- ' " H\Y : Ua]mH\YfUdmUbX'GmgHYa]WF YgYUfWk 7 YbHY \hhd.#k k k "i Y`! ZngfWcf[#
- ("GcWU`FYgYUfW\5ggcWJUhJcbkkk"h\Y!gfU"cf["i_
-) "Ei Ubhhuhj Y'F YgYUfW.'8Yg][b'\hd.#k k k "Yl dYf]a YbhfYgci fWrg"Wca #ei Ubhhuhj Y!fYgYUfW.! XYa][b"\ha`'
- *" 9j]XYbWY 6UgYX DfUWfWY]b < YUTh\ 7 UfY !! \hd.#k k k "V]ca YX"]V"i a b"YXi #YUfb#YVd
- +" Dfc[fUa '9j U`i Uh]cb'UbX'8Yg][b'!'\hd.#a UbU[Ya Ybh\Y`d"cf[#j U`i Uhb#b`SYj U`"\ha

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Hi fb]h]b Gi Va]gg]cb

#mci YbWci bhYf'U'dfcV'Ya k \Yb'UhYa dhJo['hc'gi Va]hmci f'Ugg][ba Ybhh\fci [\'Hi fb]hJbž'd'YUgY
hY'Yd\cbY'9| hYfbU'Gi ddcfhcb'-',) '' ' ' %cf'Ya U] 'h\Ya 'cb'Yl hYfbU'hY'hgi ddcfh4 i bgk "YXi "Ui '"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY_XUmg'UbX'-.\$\$Ua '-') .\$\$da 'cb'k YY_YbXg'fl *) 'XUmg'U'mYUfŁ"'#mci
UfY'i bUV'Y'hc'gi Va]hmci f'Ugg][ba YbhXi Y'hc'U'ZJi 'hk]h\'Hi fb]hJb'mci 'a Um'Udd'mZcf'Ub'Yl hYbg]cbž'Vi hmci
a i ghfYhJ]b'mci f'hJW_Yhbi a VYf'Zfca '9| hYfbU''Gi ddcfhfU'cb['k]h\'Ubmch\Yf'fY'Yj UbhXcW'a YbhgŁ'hc
]bWi XY'Ug'Yj]XYbWY'hc'gi ddcfhmci f'Yl hYbg]cb'Udd']WUhJcb"'#mci 'Ya U]''9| hYfbU''Gi ddcfhmci 'k]'`
Ui hca UhJWJ'`mfYWY']j Y'U'HJW_Yhbi a VYfž'Vi h]Zmci 'hY'Yd\cbY'mci 'k]''`bYYX'hc'gdYWJZJWJ'`mUg_'Zcf'cbY''
Hi fb]hJb'U'gc'dfcj]XYg'i dXUhYg'cb'h\Y]f'gmghYa 'gfUh' g'cb'Hk]hYf"

; YbYfU``mž'UggYgga YbhiHug_g'a i ghVY'gi Va]hhYX'Y`YVWfcb]\W)``mj]U'Y]h\Yf'Hi fb]hjb'cf'U'A ccX`Y Ugg][ba Ybh'' =b']bgHJb\W'g'k \YfY'h\]g']g'bchdcgg]V`Yž']hk]``'VY'gHJhYX'cb'mci f'\\circ i fgY'g'A ccX`Y'g]hY'k]h\ U`hYfbUhj Y'gi Va]gg]cb'XYHJ]`g''

cfi]bZcfa Uh]cbicbi\ckihcigi Va]hUgg][ba Ybhgicbi]bYij]UiA ccXiY.i\hdg.#gh XYbh'i bgk "YXi "Ui #\ck! qi Va]hUgq][ba Ybha ccXiY

$5\,WUXYa\,W< cbYghmUbXDU[\,Uf]ga$

D`U[]Uf]ga i]gii g]b[ih\Yik cfXgicfi]XYUgicZch\YfgiUbXidfYgYbh]b[ih\Ya iUgimci fick b"i=iWUbiHU_Yia Ubm Zcfa gžZfca iXYi]VYfUhYiWXYUh]b[ihciUWW]XYbhU`mWcdm]b[iZfca iUigci fWYik]h\ci hiUW_bck iYX[Ya Ybh'i



5 WUXYa]W±bZcfa Uhjcb

: cf'YggYbhJU`ghi XYbh]bZcfa UhJcb'fY`UhJb['hc. • fYei YghgʻZcfʻYl hYbg]cb/