An axiomatic theory for comonotonicity-based risk sharing

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1. Introduction

Consider a pool of individual random future losses.

Decentralized risk-sharing:

- Refers to risk-sharing (RS) mechanisms under which the participants in the pool share their risks among each other.
- Each participant in the risk-sharing pool is compensated *ex-post* from the pool for his loss.
- In return, each participant pays an ex-post contribution to the pool.
- These contributions follow from a risk-sharing rule, satisfying the *self-financing condition*.

Decentralized risk-sharing does not require an insurer, but an administrator.

2. Risk-sharing and risk-sharing rules

Agents and their losses

Let c be an appropriate (su¢ ciently rich) set of r.v.'s in the probability space $(W, \;, P)$, representing random losses at time 1.[1]

Consider n economic agents, numbered $i = 1, 2, \ldots, n$.

Each agent i faces a loss $X_i \;\; c$ at the end of the observation period $[0, 1]$.

Without insurance or pooling, each agent bears his own loss:

2. Risk-sharing and risk-sharing rules

Pools of losses

The joint cdf of the loss vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is denoted by $F_{\mathbf{X}}$.

The marginal cdf's of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$.

The aggregate loss faced by the n agents with loss vector $\mathbf{X}$ is denoted by $S_{\mathbf{X}} = \sum_{i=1}^{n} X_i$.

Hereafter, we will often call $\mathbf{X}$ the pool, and call each agent a participant in the pool.

Definition: For any pool X c^n with aggregate loss S_X the set $_n(S_X)$ is defined by:

$$ {}_n(S_X) = \left((Y_1, Y_2, \ldots, Y_n) \quad c^n \quad \sum_{i}^{n} Y_i = S_X \right) $$

The elements of $_n(S_X)$ are called the n-dimensional allocations of S_X in c^n.

2. Risk-sharing and risk-sharing rules

Risk-sharing

Definition: Risk-sharing in a pool $\mathbf{X}$ c^n is a two-stage process.

- Ex-ante step (at time 0):
  The losses X_i in the pool are re-allocated by transforming $\mathbf{X}$ into another random vector $\mathbf{H}_{\mathbf{X}}$ $_n(S_{\mathbf{X}})$:

$$\mathbf{H}_{\mathbf{X}} = H_{\mathbf{X},1}, H_{\mathbf{X},2}, \ldots, H_{\mathbf{X},n}$$

- Ex-post step (at time 1):
  - Each participant i *receives from the pool* the realization of his loss X_i.
  - In return, he *pays to the pool* a contribution equal to the realization of his re-allocated loss $H_{\mathbf{X},i}$.

Remark: As $\mathbf{H}_{\mathbf{X}}$ $_n(S_{\mathbf{X}})$, risk sharing is self-financing:

$$\sum_{i=1}^{n} H_{\mathbf{X},i} = \sum_{i=1}^{n} X_i$$

De...nition: A risk-sharing rule is a mapping $\mathrm{H} : c^n \quad c^n$ satisfying the self-...nancing condition:

$$\mathrm{X} \quad \mathrm{H}_{\mathrm{X}} \quad {}_n(S_{\mathrm{X}}), \qquad \text{for any } \mathrm{X} \quad c^n$$

Remarks: For any participant i in the pool $\mathrm{X} = (X_1, \ldots, X_n)$,

X_i is called his loss, (paid by the pool).

$H_{\mathrm{X},i}$ is called his contribution, (paid to the pool).

Contribution vector:

$$\mathrm{H}_{\mathrm{X}} = \; H_{\mathrm{X},1}, H_{\mathrm{X},2}, \ldots, H_{\mathrm{X},n}$$

Notation

2. Risk-sharing and risk-sharing rules

Aggregate risk-sharing rules

De…nition: H : c^n c^n is an aggregate RS rule if there exists a function $\mathrm{h}^{\mathrm{aggr}}$: R (c^n) R^n such that the contribution vector H_X of any X c^n is given by:

$$\mathrm{H}_\mathrm{X} = \mathrm{h}^{\mathrm{aggr}}\left(S_\mathrm{X}, F_\mathrm{X}\right)$$

Property: Any aggregate RS rule H is internal, with internal function h satisfying:

$$\mathrm{h}\left(\mathrm{X}; F_\mathrm{X}\right) = \mathrm{h}^{\mathrm{aggr}}\left(S_\mathrm{X}, F_\mathrm{X}\right) \qquad \text{for any X} \quad c^n$$

Definition: $\mathrm{H} : c^n \quad c^n$ is a dependence-free RS rule if there exists a function $\mathrm{h}^{\text{dep-free}} : \mathrm{R}^n \quad (\quad (c))^n \quad \mathrm{R}^n$ such that the contribution vector H_{X} of any $\mathrm{X} \quad c^n$ is given by:

$$\mathrm{H}_{\mathrm{X}} = \mathrm{h}^{\text{dep-free}}\left(\mathrm{X}, F_{X_1}, \ldots, F_{X_n}\right)$$

Property: Any dependence-free RS rule H is internal, with internal function h satisfying:

$$\mathrm{h}\left(\mathrm{X}; F_{\mathrm{X}}\right) = \mathrm{h}^{\text{dep-free}}\left(\mathrm{X}, F_{X_1}, \ldots, F_{X_n}\right) \qquad \text{for any } \mathrm{X} \quad c^n$$

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

Definition[2]: The conditional mean RS rule $\mathbf{H}^{cm}$ is defined by

$$H_i^{cm}(\mathbf{X}) = E\left[X_i \mid S_{\mathbf{X}}\right], \qquad i = 1, 2, \ldots, n,$$

for any $\mathbf{X} \in c^n$.

Interpretation: Each participant contributes the expected value of the loss he brings to the pool, given the aggregate loss experienced by the pool.

Property:

$\mathbf{H}^{cm}$ is internal and aggregate, but not dependence-free.

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4. The quantile risk-sharing rule

Motivation

Let $\mathrm{x} = (x_1, x_2, \ldots, x_n)$ be the realization of X.

There exist probabilities $p_1, \ldots, p_n$ such that

$$\mathrm{x} = \quad F_{X_1}{}^{1},$$

4. The quantile risk-sharing rule

Definition:

Under the quantile RS rule $\mathrm{H}^{\mathrm{quant}} : c^n \quad c^n$, the contribution vector of $\mathrm{X} \quad c^n$ is given by

$$\mathrm{H}^{\mathrm{quant}}_{\mathrm{X}} = \mathrm{h}^{\mathrm{quant}}\left(S_{\mathrm{X}}, F_{\mathrm{X}}\right)$$

where $\mathrm{h}^{\mathrm{quant}} : \mathrm{R} \qquad (c^n) \qquad \mathrm{R}^n$ is defined by

$$h_i^{\mathrm{quant}}\left(s, F_{\mathrm{X}}\right) = F_{X_i}^{\ 1}\ \ F_{S^c_{\mathrm{X}}}(s)\ , \qquad i = 1, 2, \ldots, n$$

Properties:

$\mathrm{H}^{\mathrm{quant}}$ satisfies the self-financing condition.

$\mathrm{H}^{\mathrm{quant}}$ is an aggregate RS rule.

$\mathrm{H}^{\mathrm{quant}}$ is a dependence-free RS rule.

5. The 'stand-alone for comonotonic pools' property

De…nition: X c^n is a comonotonic pool in case

$$\mathrm{X} \stackrel{\mathrm{d}}{=} \left(F_{X_1}^{\ 1}(U), \ldots, F_{X_n}^{\ 1}(U) \right)$$

De…nition: A RS rule H : c^n c^n satis…es the stand-alone for comonotonic pools property if for any comonotonic pool X^c c^n, one has that

$$\mathrm{H}_{\mathrm{X}^c} = \mathrm{X}^c$$

Property: $\mathrm{H}^{\mathrm{quant}}$ satis…es the stand-alone for comonotonic pools property.

6. Axiomatic characterization of the quantile RS rule

Theorem:

Consider the internal RS rule $H : c^n \to c^n$.

H is the quantile RS rule if, and only if, it satisfies the following axioms:

(1) H is aggregate.

(2) H is dependence-free.

(3) H is (generalized) stand-alone for comonotonic pools[4].

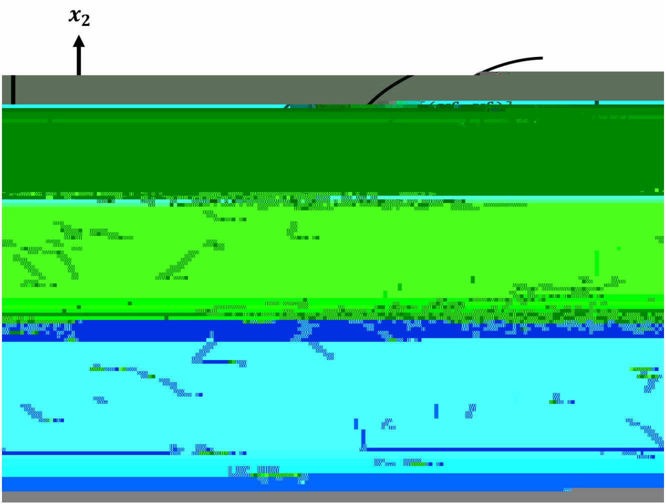
Proposition:

The axioms (1), (2) and (3) are independent.

[4] The 'generalized stand-alone for comonotonic pools' property is a slightly stronger property than the 'stand-alone for comonotonic pools' property, see D,R,C,D (2023).

6. Axiomatic characterization of the quantile RS rule

Graphical proof of the theorem (bivariate case)



$$h\left(x\ \ , F_X\right) \overset{\text{axiom 1}}{=} h\left(x^c, F_X\right) \overset{\text{axiom 2}}{=} h\left(x^c, F_{X^c}\right) \overset{\text{axiom 3}}{=} x^c$$

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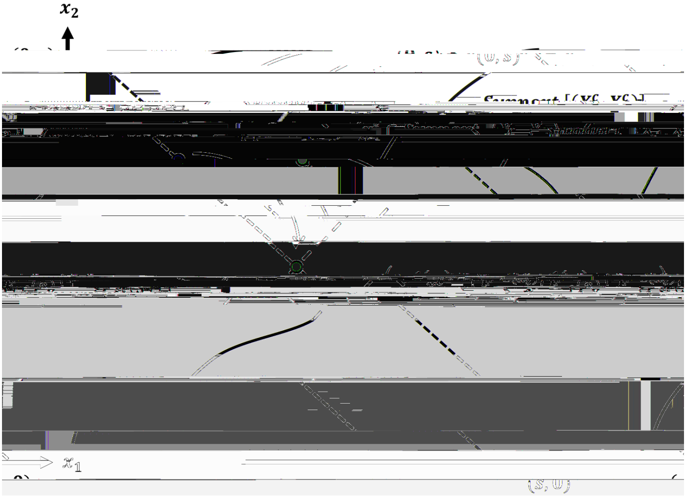

$$h(x, F_X) \overset{\text{axiom 1}}{=} h(x^c, F_X) \overset{\text{axiom 2}}{=} h(x^c, F_{X^c}) \overset{\text{axiom 3}}{=} x^c$$

7. Example of a non-internal risk-sharing rule

Consider the RS rule H : c^n c^n, where any X c^n is a pool of health-related costs of the participants.

References

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