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2010 University of New South Wales School Mathematics Competition [1]

Junior Division – Problems and Solutions

Problem 1

Find the set of all pairs of positive integers (n) that satisfy

$$|n^2 - \quad ^2 - 2010| \leq 1$$

Solution 1

We being by factoring then we seek n and that satisfy one of the following:

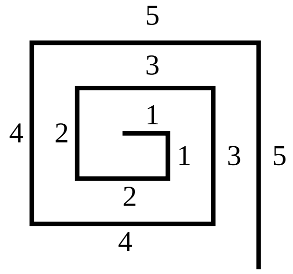
(i) $(n - \quad)(n + \quad) = 2009 = (1)(7)(7)(41)(2009)$ or

(ii) $(n - \quad)(n + \quad) = 2010 = (1)(2)(3)(5)(67)(2010)$ or

(iii) $(n - \quad)(n + m$

$$
\begin{aligned}
x(x+1)(x+2)(x+3)+1 &= [(x+1)(x+2)][(x(x+3)]+1 \\
&= (x^2+3x+2)(x^2+3x)+1 \\
&= [(x+\frac{3}{2})^2 - \frac{1}{-}
\end{aligned}
$$

A monk sets out from a monastery in a valley at dawn and follows a winding path up a mountainside at a constant speed, planning to arrive at a temple on the mountain-top at dusk. A second monk sets out from the temple at dawn and travels down the mountainside along the same path, but at twice the speed, until she meets the monk coming up and then they stop for a break together. The temple is at an elevation 945 metres above the elevation of the monastery. When viewed from above the winding path appears as a regular rectangular spiral with the geometry of the central portion as shown below.



The two shortest segments of this spiral have length of 1 metre each and the two

3. Suppose that the two monks meet on a spiral arm segment of length . Then we require either i) $2(1 + 2 + \quad - 1 \quad) + \quad 6600$ and $2(1 + 2 + \quad) \quad 6600$ or ii) $2(1 + 2 + \quad) \quad 6600$ and $2(1 + 2 + \quad) + (\quad + 1) \quad 6600$ for some integer . Note that $2(1 + 2 + 80 \quad) = 6480 \;\; 2(1 + 2 + 80 \quad) + 81 = 6561 \;\; 2(1 + 2 + 81 \quad) = 6642$ so that condition i) is satisfied for $\quad = 81$. The two monks meet on a spiral arm of length 81 metres, when viewed from above.

An alternate method of solution using trigonometry results in a simpler solution

A cubic block can be partitioned into smaller cubic blocks in many ways. An integer n is called a cute-cube number if a cubic block can be partitioned into n cubic blocks of at most two different sizes.

1. Provide an example of a cute-cube number that is greater than 2^3 but less than 3^3.
2. Show that 2010 is a cute-cube number.

Solution 4

2. More generally, starting with N we construct N with digits $b = 9 \geq b_2 \geq b_3$ and then the digits of N_2 are $10 + b_3 - b$ 9 $b - b_3 - 1 = b_3 + 1$ 9 $8 - b_3$. The largest digit is 9 and the sum of the remaining digits is 9. Note too that the smallest digit is increased by one and thus the larger is decreased by one. This can only continue until the input is 495 which must occur after at most six steps.

Problem 6

Let (n) denote the number of positive factors of a positive integer n. Prove that, for any positive integers and n, $(n) \leq ( )\ (n)$.

Solution 6

Let where $i = 1\ 2\ 3$ denote the prime factors of and the prime factors of n (some of which may be common). Using the product symbol $\prod_i^K x_i$ to denote the product $x \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_K$ we can write

$$= \prod_i^K p_i^{\alpha} \qquad {}_i \geq 0$$

$$n = \prod_i^K p_i^{\beta} \qquad {}_i \geq 0$$

$$n = \prod_i^K p_i^{\alpha \quad \beta}$$

The divisors of p_i^{δ} are 1 p_i p_i^{δ} so that $(p_i^{\delta}) = {}_i + 1$ and

$$( ) = \prod_i^K ({}_i + 1)$$

$$(n) = \prod_i^K ({}_i + 1)$$

$$(n) = \prod_i^K ({}_i + {}_i + 1)$$

The result (n) $( )\ (n)$ now follows since

$$({}_i + {}_i + 1) \leq ({}_i {}_i + {}_i + {}_i + 1) = ({}_i + 1)({}_i + 1)$$

An alternate proof is possible without using unique factorisation into primes. This alternate proof starts with the proposition that if d is a divisor of (i.e. $d|\ n$) then $d = d\ d_2$ where $d\ |$ and $d_2|n$. It then follows that

$$\begin{aligned} (n) &= |\{d \square \mathrm{Z} \ : d| \ n\}| \\ &= |\{d\ d_2 : d \ d_2 \square \mathrm{Z} \ \ d\ | \ \ d_2|n\}| \\ &\leq ( )\ (n) \end{aligned}$$

Of course it remains to prove the proposition (see the problems section in this issue).

Senior Division – Problems and Solutions

Problem 1
See Problem 6 in the Junior Competition.

Solution 1
See Problem 6 solution in the Junior Competition.

Problem 2
The infinite order tower power of x is defined as

$$T(x) = x^{x^{x^{x^{\cdot^{\cdot}}}}} = x^{x^{(x^{(x^{\cdot\cdot})})}}$$

1. Find the largest number x for which $T(x)$ is finite.
2. Find the value of $T(x)$ in this case.

Solution 2
First we may note that $T(1) = 1$ and thus we seek $x_{max} \geq 1$. Let

$$T(x) = x^{x^{x^{x^{\cdot^{\cdot}}}}}$$

then take logarithms of each side to obtain

$$\log T = \log x^{T} = T \log x$$

Now solve for

$$x(T) = \exp\left(\frac{\log T}{T}\right)$$

To find the maximum of $x(T)$ differentiate x with respect to T then

$$\frac{dx}{dT} = \left(\frac{1 - \log T}{T^2}\right) \exp\left(\frac{\log T}{T}\right)$$

Now dx

(i) a unique solution in t for $0 \quad x \leq 1$

(ii) two solutions in t for $1 \quad x \quad e^{\frac{1}{\ }}$

(iii) one solution in t for $x = e^{\frac{1}{\ }}$

(iv) no solutions in t for $x \quad e^{\frac{1}{\ }}$

Then taking $x = e^{\frac{1}{\ }}$ we have $e^{\frac{t}{\ }} = t$ if $t = e$.

Problem 3

A gaoler enters a room with three prisoners and places ten hats in clear view on a table in front of the prisoners. Some of the hats are black and the others are white. The gaoler blindfolds the prisoners and then puts a hat on each of them and removes the remaining seven hats and says, "I will give you turns to deduce the colour of the hat that I have put on your head. If you can do this correctly you will be set free."

He then removes the blindfold from the first prisoner who says, "I can see the

The remaining possibilities all have a white hat on Prisoner 3.

If there were three, four, five, six or seven black hats then in addition to the above there would be the possibility of a black hat on each prisoner in which case Prisoner 3 could not deduce if he had black or white.

If there were eight black hats then there were only two white hats and by swapping black and white in the above table Prisoner 3 would have been led to the conclusion that he had a black hat but this was not his conclusion.

If there was only one white hat then similar to the case of one black hat, Prisoner 3 would have been able to deduce the colours of the hats of all three.

The case of no white hats is trivial.

Problem 4

Consider a triangle with sides a b c of unequal length, $a \quad b \quad c$. Construct a sequence of triangles $T \quad T_2$ as follows:

Let $s = \dfrac{a + c}{2}$ and let T have sides $s \quad s \quad b$.

Let $s_2 = \dfrac{s + b}{2}$ and let T_2 have sides $s_2\, s_2\, s$.

Let $s_3 = \dfrac{s_2 + s}{2}$ and let T_3 have sides $s_3 \; s_3 \; s_2$

For $n \geq 3$, let $s_n = \frac{}{2}(s_{n-} + s_{n-2})$ and let T_n have sides $s_n \; s_n \; s_{n-}$.

1. Prove that each triangle in the sequence has perimeter $a + b + c$.
2. Prove that for $n \geq 3$, $s_n - s_{n-} = \dfrac{(-1)^{n-}}{2^{n-}}(s - b)$
3. What happens to the three sides of T_n as n increases without bound?

Solution 4

1. Let $P(n)$ be the proposition $2s_n + s_{n-} = a + b + c$. We also define $s = b$. Clearly $P(2)$

2.

$$
\begin{aligned}
s_n - s_{n-1} &= \frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1} \\
&= (-\frac{1}{2})(s_{n-1} - s_{n-2}) \\
&= (-\frac{1}{2})^2(s_{n-2} - s_{n-3}) \\
&\vdots \\
&= (-\frac{1}{2})^{n-2}(s_2 - s_1) \\
&= (-\frac{1}{2})^{n-2}(s_2 - b)
\end{aligned}
$$

3. [illegible] ∞, [illegible] i[illegible]nite[illegible]o that[illegible]he sides [illegible] o[illegible]
[illegible]e $T(n)$ [illegible]o[illegible]s, th[illegible]ng[illegible]le is equi[illegible]

P[illegible]

Problem 6
Consider the list of fractions

$$\frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{20}$$