MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 12 Solutions, August 27, 2019

1. Since $1! + 2! + 3! = 9 = 3^2$, we know that $n = 3$ is a possible solution to the problem. We will show that $n = 3$ is the largest solution. Observe that for any number to be a perfect square, it cannot end in the digit 3. Since $1! + 2! + 3! + 4! = 33$, $n = 4$ is not a solution. Moreover, $n!$ contains factors of both 2 and 5 for $n > 4$, therefore $n!$ ends in the digit 0 for $n > 4$. We can now conclude that the number $1 + 2! + 3! + \dots + (n-1)! + n!$ ends in the digit 3 for $n > 4$, thus cannot be a perfect square.

2. Since we are adding consecutive integers, we know that $a_2 = a_1 + 1; a_3 = a_1 + 2; \dots; a_{100} = a_1 + 99$. Therefore, we can write

$$\begin{aligned} \sqrt{a_2 + a_3 + \dots + a_{99}} - \sqrt{a_1 + a_{100}} &= \sqrt{98a_1 + 1 + 2 + \dots + 98} - \sqrt{2a_1 + 99} \\ &= \sqrt{98a_1 + 4851} - \sqrt{2a_1 + 99}. \end{aligned}$$

   The second line of the equation above is minimal when $a_1 = 1$.

3. Let abc and efg be three digit numbers. Then we can write the initial 6-digit number x as

$$x = 1000 \cdot abc + efg.$$

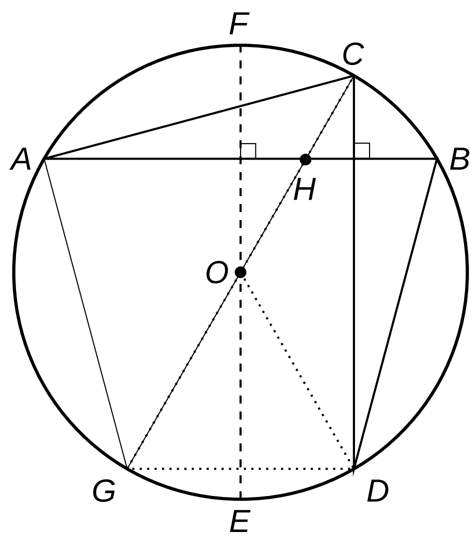
   We also know that

$$6x = 1000 \cdot efg + abc.$$

   Combining the above two equations gives $5999 \cdot abc = 994 \cdot efg$, which can be further simplified to $857 \cdot abc = 142 \cdot efg$. Hence the number we are after is 142857.

4. To have all f7461 -20.96.7142881us1!1!e94326(are)-3 [1(r)atr1!1!vraI2727(e)-380(all)-380(f7461 -20.96

5. Let EF be the diameter perpendicular to CD, and let G be the reflection of D in the line EF, as shown below.



Since G is the reflection of D, $AG = BD$. If we can show that COG is a diameter of the circle, then $\triangle ACG$ is right angled (by Thales' theorem) and the desired result follows by Pythagoras' theorem. In order to show that COG is a diameter of the circle, we must show that $\angle COG$ is straight.

Let H be the point of intersection of OC ad AB. Let $\angle OCD =$ and $\angle CHB =$. Since $CD \perp AB$, and are complementary. Furthermore, $\angle CHB$ and $\angle OHA$ are vertically opposite, so $\angle OHA =$ a nd thus $\angle HOF =$.

Now $\triangle OCD$ is isosceles, thus $\angle ODC = \angle OCD =$, and since FE is parallel to CD, $\angle EOD = \angle ODC =$. As G is the reflection of D in the line EF, $\triangle OEG$ $\triangle OED$, and thus $\angle GOE = \angle DOE =$. Thus $\angle GOE$ and $\angle HOF$ are equal and thus vertically opposite, and so $\angle GOC$ is straight, as required.