[redacted]

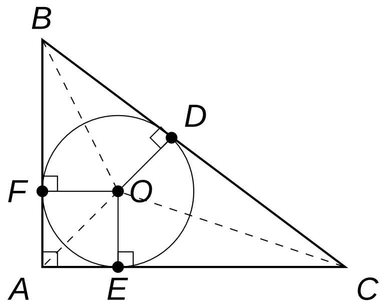
# MATHEMATICS ENRICHMENT CLUB.
## Solution Sheet 7, June 25, 2019[1]

1. Note that $x$ has greater magnitude than $y$. Firstly, let's concentrate on positive solutions to the equation. If $2019 = x^2 - y^2$, then $2019 = (x + y)(x - y)$. The factors of 2019 are 1, 3, 673, and 2019. So

| $(x - y)$ | $(x + y)$ | $x$ | $y$ |
|---|---|---|---|
| 1 | 2019 | 1010 | 1009 |
| 3 | 673 | 338 | 335 |

Thus the solutions are $(1010, \pm 1009)$, $(-1010, \pm 1009)$, $(338, \pm 335)$ and $(-338, \pm 335)$, so there are eight solutions altogether.

2. Let $O$ be the centre of the incircle, let the radius of the incircle be $r$ and let $D$, $E$ and $F$ be the points of tangency between the incircle and the triangle as shown below.




Since $OD$, $OE$, and $OF$ are radii to tangents, $\angle BFO = \angle CEO = \angle ODB = 90^\circ$. Thus $AFOE$ is a square with side length $r$. Hence $AE = AF = r$, $EC = b - r$ and $FB = c - r$. Furthermore, by RHS, $\triangle EOC \cong \triangle DOC$ and thus $DC = b - r$. Similarly, $BD = c - r$. Thus

$$a = (b - r) + (c - r)$$
$$r = \frac{1}{2}(b + c - a)$$

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

3. A neat trick is to express $N$ as

$$\underbrace{333\cdots333}_{61\ 3's} = \frac{3}{9}\left(\underbrace{999\cdots999}_{61\ 9's}\right) = \frac{1}{3}(10^{61}-1).$$

Similarly, $M = \underbrace{666\cdots666}_{62\ 6's} = \frac{2}{3}(10^{62}-1)$. Now

$$\begin{aligned} N \times M &= \frac{2}{9}(10^{61}-1)(10^{62}-1) \\ &= \frac{2}{9}(10^{61}-1) \times 10^{62} - \frac{2}{9}(10^{61}-1) \\ &= \underbrace{222\cdots222}_{60\ 2's}\underbrace{000\cdots000}_{62\ 0's} - \underbrace{222\cdots222}_{60\ 2's} \\ &= \underbrace{222\cdots222}_{60\ 2's}19\underbrace{777\cdots777}_{60\ 7's}8. \end{aligned}$$

4. In modular arithmetic, if $a \equiv b \bmod (n)$, then $a^x \equiv b^x \bmod (n)$. Thus we can see that

$$\begin{aligned} a &\equiv 1 \bmod (a-1) \\ a^x &\equiv 1^x \bmod (a-1) \\ &\equiv 1 \bmod (a-1) \end{aligned}$$

Similarly,

$$\begin{aligned} a &\equiv a \bmod (a+1) \\ &\equiv (-1) \bmod (a+1) \\ a^x &\equiv (-1)^x \bmod (a+1) \end{aligned}$$

$$(-1)^x \bmod (a+1) \equiv 1 \bmod (a+1) \quad a$$

Thus $r_1 + r_2 = a + 1$.

5. We can write $x = n + d$, where $n$ is the integral part of $x$ and $d$ the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of $d$; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

| $[2x]$ | + | $[4x]$ | + | $[6x]$ | + | $[8x]$ | | |
|---|---|---|---|---|---|---|---|---|
| 0 | + | 0 | + | 0 | + | 1 | $= 1;$ | if $\frac{1}{8} \le d < \frac{1}{6}.$ |
| 0 | + | 0 | + | 1 | + | 1 | $= 2;$ | if $\frac{1}{6} \le d < \frac{1}{4}.$ |
| 0 | + | 1 | + | 1 | + | 2 | $= 4;$ | if $\frac{1}{4} \le d < \frac{1}{3}.$ |
| 0 | + | 1 | + | 2 | + | 2 | $= 5;$ | if $\frac{1}{3} \le d < \frac{3}{8}.$ |
| 0 | + | 1 | + | 2 | + | 3 | $= 6;$ | if $\frac{3}{8} \le d < \frac{1}{2}.$ |

If we continue with the above calculations, the results are the numbers ending in 3; 7; 8 or 9 can not be expressed in the form $[2x] + [4x] + [6x] + [8x]$. This means that, for $n = 0$, we have 0 (in this case we can't actually count this one, as we are looking at positive integers), 1, 2, 4, 5 and 6. For $n = 1$, we have 20, 21, 22, 24, 25 and 26 (6 possibilities). For $n = 2$, we have 40, 41, 42, 24, 45 and 46, and so on. Since we are also counting 1000 itself, there are a total of 300 numbers that can be written this way.

6. Let $d$ be the number of kilometres travelled before the tyre switch is made. Then $\frac{d}{x}$ is the proportion of wear on the front tyre before the switch, hence they will travel a further $1 \quad \frac{d}{x} \; y$ kilometres before the tyres are retired. So the total distance travelled by the font tyre is $d + \; 1 \quad \frac{d}{x} \; y$. Similarly, the total distance travelled by the rear tyre is $d + \; 1 \quad \frac{d}{y} \; x$.

   Suppose the claim of the advertisement is true, then we must have the following system of inequalities

$$\begin{array}{l} d + \; 1 \quad \frac{d}{x} \; y \quad \frac{x+y}{2} \\ d + \; 1 \quad \frac{d}{y} \; x \quad \frac{x+y}{2} \end{array} :$$

   Rearranging this gives

$$\begin{array}{l} d \;\; 1 \quad \frac{y}{x} \qquad \frac{x \;\; y}{2} \\ d \;\; 1 \quad \frac{x}{y} \qquad \frac{y \;\; x}{2}; \end{array}$$

   then using the assumption that $x < y$, we have

$$d \quad \frac{x \;\; y}{2} \qquad 1 \quad \frac{y}{x}$$

## Senior Questions

1. Since $> 0$, $+ \frac{1}{}^{2} = {}^{2} + \frac{1}{{}^{2}} + 2 \quad 2$. Similarly, $+ \frac{1}{}^{2} \quad 2$. Therefore, if $r_1$ and $r_2$ are the roots of $f$ (assuming $r_1 \quad r_2$ wlog), then $r_1 \quad 2$ and $r_2 < 0$, so that $r_1 r_2 = c \quad 3 < 0$, which implies $c < 3$.

   To get the lower bound on $c$, we use the quadratic formula $2 \quad r_1 = (c + 1) + \sqrt{(c + 1)^2 \quad 4(c \quad 3)}$. Solving gives $2 \quad c$.

2. Square both sides of the equation $\sqrt{a} \quad b = \sqrt{c}$ and rearranging gives

$$\sqrt{c} = \frac{a \quad b^2 \quad c}{2b}:$$

   Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x{=}y$, where $x$ and $y$ are integers with greatest common multiplier one. Then $c = x^2{=}y^2$, and greatest common multiplier between $x^2$ and $y^2$ is one. Since $c$ is an integer, $x^2$ must be divisible by $y^2$, which can only happen if $y^2 = 1$, because the greatest common multiplier between $x^2$ and $y^2$ is one. Hence $c = x^2$, so that $c$ is a perfect square.

   If $c$ is a perfect square, then the equation $\sqrt{a} \quad b = \sqrt{c}$ implies that $a$ is also a perfect square.

3. Use the method of re ection. Re ect the point $B$ in the line that represents the river bank. This is shown as $B^0$ in the diagram below. Then the shortest distance from $A$ to $B^0$ is clearly a straight line. We can use Pythagoras' theorem to show that this is 15 km.