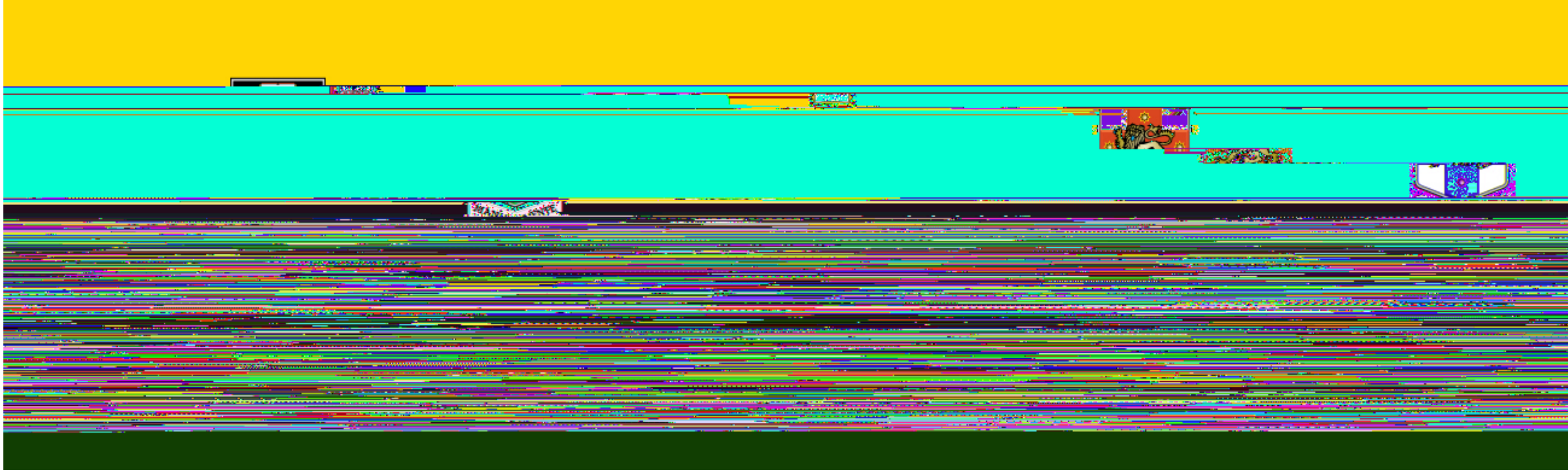

MATHEMATICS ENRICHMENT CLUB.[1]

Solution Sheet 11, August 6, 2013

1. (a)

$$\begin{aligned} 0 &\le (a - b)^2 \quad \text{with equality only if } a = b \\ 0 &\le a^2 + b^2 - 2ab \\ ab &\le \frac{a^2 + b^2}{2} \end{aligned}$$

so ab is largest when $a = b$, and since $a + b = k$ then at $a = b = \frac{k}{2}$.

(b) From above, first note that $xy \le \frac{c^2}{2}$, then

$$\begin{aligned} c^4 &= (x^2 + y^2)^2 \\ c^4 &= x^4 + y^4 + 2x^2y^2 \\ x^4 + y^4 &= c^4 - 2x^2y^2; \end{aligned}$$

which is minimum when x^2y^2 is maximum, which from above is when $x = y$ and has a value of $\left(\frac{c^2}{2}\right)^2$. So the minimum value of $x^4 + y^4 = c^4 - \frac{c^4}{2} = \frac{c^4}{2}$.

2. Construct the triangles APB and AQB. Let P' be at the intersection of the circle and the line AP, now since AB is a diameter and P' on the circle, triangle $AP'B$ is right at P', which also means triangle $PP'B$ is right at P', and so $\angle APB$

(b) By similar logic, both and must be even (to have even s[redacted]ven product). If they are both even, then the product must be divisibl[redacted]rite $= 2m$ $= 2n$ then $= 4nm$), but 2 382 982 is not, and hence [redacted]no integer solutions.

4. To make \$10 out of n 50c coins and m 20c coins we must satisfy

$$5n + 2m = 100; \quad n; m\ 2\ \mathrm{Z}; \ n; m > 0$$

or

$$m = 100 \quad 5\frac{n}{2}; \quad m; n\ 2\ \mathrm{Z}; \ n; m > 0:$$

So we merely count the number of n which are divisible by 2 and satisf[redacted]ve, of which there are 9.

5. (a) In general, if we prime factorise $x = p_1^{m_1} p_2^{m_2} \quad p_k^{m_k}$ then every [redacted]n be written as $p_1^{a_1} p_2^{a_2} \quad p_k^{a_k}$ where each $a_= 0; 1; :::; m_i$. So there are [redacted]hoices for a_1, $m_2 + 1$ choices for a_2 and so on, and hence the numbe[redacted]ors is $(m_1 + 1)(m_2 + 2) \quad (m_k + 1)$. Then $20 = 2^2 \quad 5$ and so has 3 2[redacted]rs, so (20) = 6.

If $n = p_1^{m_1} \quad p_k^{m_k}$, then $n^2 = p_1^{2m_1} \quad p_k^{2m_k}$ and so $(n^2) = (2$[redacted]$2m_2 + 1) \quad (2m_k + 1)$ which is a product of odd numbers and hence o[redacted] cannot be equal to the even number 2 (n).

(b) The number $144^2 = (3 \quad 2^2)^4 = 3^4 \quad 2^8$ so [redacted]n)3311.95[redacted]

Now if $n^4 - 6n^3 - 18n^2 + 6n + 1$ is prime its only factors are itself and 1. Since

$$n^4 - 6n^3 - 18n^2 + 6n + 1 = (n^2 - 3n - 1 - 5n)(n^2 - 3n - 1 + 5n) = (n^2 - 8n - 1)(n^2 + 2n - 1)$$