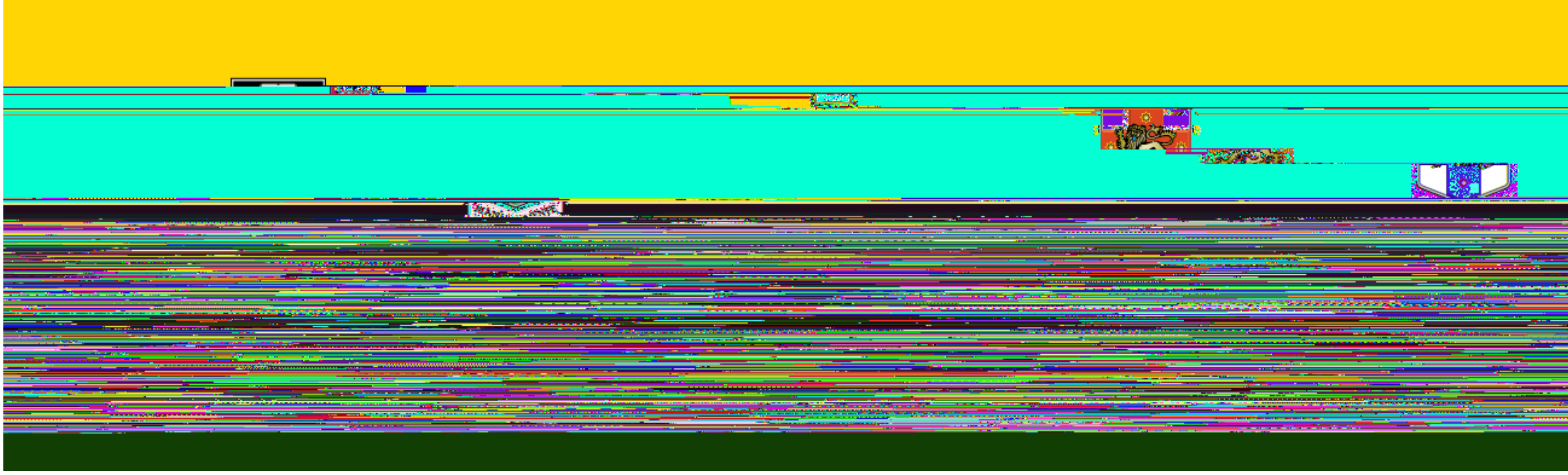

MATHEMATICS ENRICHMENT CLUB.[1]

Solution Sheet 4, May 28, 2013

1. (a) Writing $(21)_b$ and $(12)_b$ in base ten then we must have $2b+1 = 2(b+2) \implies 1 = 4$, a contradiction.

   (b) In base ten we must satisfy $ab + c = 2(cb +$

Writing $b = 3m + 2$ we can replace all bs with ms and get

$$a = (2m+1)k$$
$$c = mk$$
$$k < \frac{3m+2}{2m+1}, \quad k \in \mathbb{N}, \; m \in \mathbb{N}.$$

2. Write $1000 = \sum_{k=a}^{b} (2k-1)$, $0 < a \le b$ which is an arithmetic progression, so reduces to $1000 = (b-(a-1))(b+(a-1))$. So now we look for two numbers $x = b, y = a-1$ whose sum and difference are both factors of 1000. The factors of 1000 are $(1, 1000)$, $(2, 500)$, $(4, 250)$, $(8, 125)$, $(10, 100)$, $(20, 50)$, $(25, 40)$. Since we must have one factor represented by $x - y$ and the other by $x + y$, both factors must be even, which leaves 4 possible pairs for x and y, and hence 4 pairs b and a (since $a \le b$).

   Finally, if 1000 is the sum of consecutive, positive odd numbers $k + (k+2) + (k+4) + \cdots$, we can also add all the odd integers $-(k-2), -(k-4), \ldots, -3, -1, 1, 3, \ldots, (k-2)$ without changing the original value. So for each of the 4 pairs a and b above, we have another representation. Hence, I count 8 ways.

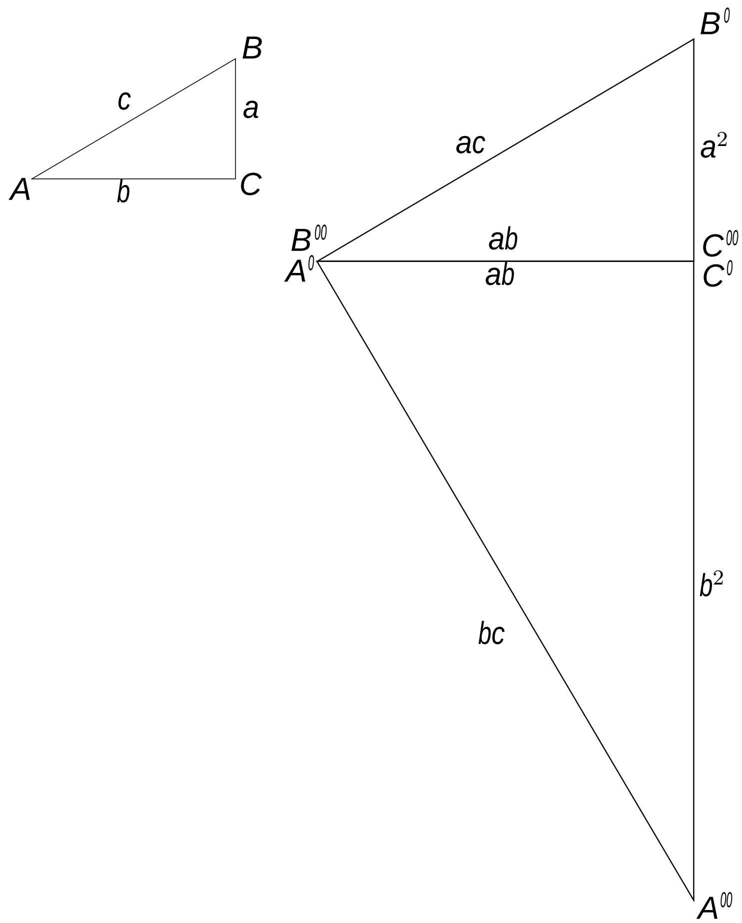

Figure 1: Picture for question 3

3. The new triangle is $A'''B'''C'''$, where C'''

triangle $A'''B'''C'''$ is the enlargement of ABC by a factor of c, which implies $a^2+b^2 = c^2$; Pythagorus' Theorem.

4. Angle A is 10 and angle C is 30 .

5. (a) The triangles BAD and KAL are similar since they have two sides in ratio ($AK : AB = 1 : 3$ and $AL : AD = 1 : 3$) which contain the common angle A. For the same reasons, trianlges BCD and NCM are similar. Thus KL is parallel to BD which is parallel to MN. Also, the lengths $BD = 3KL$ and $BD = 3MN$ so $KL = MN$. Thus $KLMN$ is a parallelogram because it has one pair of equal length and parallel sides.

   (b) In the same fashion as above we show that the triangles ABC and KBN are similar, and that the trianlges ADC and LDM are similar, with $AB : KB = 3 : 2$, $BC : BN = 3 : 2$, $AD : LD = 3 : 2$ and $DC : DM = 3 : 2$. Thus the areas are in the ratios area(ABD) : area(AKL) = 1 : 9, area(BCD) : area(NCM) = 1 : 9, area(ABC) : area(KBN) = 4 : 9 and area(ADC) : area(LDM) = 4 : 9. From this we obtain the two equations

$$\text{area}(ABCD) = \text{area}(ABD) + \text{area}(BCD) = 9\,\text{area}(AKL) + 9\,\text{area}(NCM)$$
$$\text{area}(ABCD) = \text{area}(ABC) + \text{area}(ADC) = \frac{9}{4}\,\text{area}(KBN) + \frac{9}{4}\,\text{area}(LDM);$$

   combining which yields

$$\begin{aligned}\text{area}(ABCD) + 4\,\text{area}(ABCD) &= 9(\text{area}(AKL) + \text{area}(NCM) + \text{area}(KBN) + \text{area}(LDM)\\ 5\,\text{area}(ABCD) &= 9(\text{area}(ABCD) \quad \text{area}(KLMN))\\ 9\,\text{area}(KLMN) &= 4\,\text{area}(ABCD)\\ \text{area}(KLMN) &= \frac{4}{9}\,\text{area}(ABCD).\end{aligned}$$

6. Consider a right triangle with perpendicular sides of length m and n, and thus hypotenuse $\sqrt{m^2+n^2}$. Say one of the non-right angles is then

$$\frac{m+n}{\sqrt{m^2+n^2}} = \cos \quad + \sin \; .$$

   The value cos + sin is maximum when cos = sin , and so when cos = sin = $\frac{1}{\sqrt{2}}$. Thus

$$\frac{m+n}{\sqrt{m^2+n^2}} \quad \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Senior Questions

1. The radii of the inscribed circle that touch the triangle divide the triangle in to 2 pairs of congruent triangles and a square. Thus the perimeter of the triangle is $p = 2x+2y+2$ 2 where $x + y = 15$ cm is the length of the hy-34.436 Td [(6.)]TJ0 gn0.9550(arm)]TJ/Fils]TJ76(15)]T