MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 6, June 11, 2013

1. The prime factorisation of 770 = 2 5 7 11, so assuming by adults we mean over 18 year olds, our two people are 22 and 35.

2. (Disclaimer: Introduction `group theory' answer - this question can be answered more simply by deductive logic, or guess and check (maximum 13 guesses), but this question's close ties to group theory I think warrants a bit of abstract algebra. If you just want the answer, skip to the end©)

   Let's write the card shu er as a function , where (n) is the new position of thenth card after one shu e. We'll also write iterated shu es as ^{m}, meaningm compositions of the shu ing function . As a nal piece of notation, we'll introduce k-cycles', which are written as a collection of numbers in a pair of brackets and indicate that the value of each number is that to its immediate right (or the rst position if at the end of the cycle), e.g. (1 2 3) means 1! 2, 2! 3 and 3! 1.

   The information given tells us

   $$^2 = (1\ 12\ 5\ 2\ 7\ 9\ 11\ 10\ 4\ 13\ 3\ 8\ 6)$$

   We can multiply (compose) cycles together just by tracing from left (i.e. applying the cycles to each number left to right), for example

   $$(1\ 2\ 3)(2\ 1\ 4) = (1)(2\ 3\ 4) = (2\ 3\ 4)$$

   since 1! 2! 1, 2! 3, 3! 1! 4 and 4! 2.[2] In this manner we can repeatedly multiply ^{2} and we nd

   $$^{26} = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)\ ;$$

   i.e. shu ing 26 times puts the cards back in to the order they originally were. This means the `order' of is 26, where the `order' of a permutation is how many times

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

[2]An interesting result is that every permutation can be written as a product of 2-cycles, e.g. (1 2 3) = (1 3)(3 2), and even though this 2-cycle representation is not unique, it is always made up of either an odd or even number of 2-cycles.

you multiply it by itself to get the identify function - one that leaves everything alone like the one above.

Since is, at most, a 13-cycle its order is 13. So the order of could be 1; 2; 13 or 26 in order to satisfy ^{26} = (), but it can't be 26, it's not 1 or 2 from the given information, so it must have order 13.

So now we work out ^{12}, then we can determine so that ^{12} = (). I worked out ^{12} by rst performing

$$^{2}\ ^{2} = \ ^{2} = (1\ 5\ 7\ 11\ 4\ 3\ 6\ 12\ 2\ 9\ 10\ 13\ 8)$$

then

$$^{8} = \ ^{4}\ ^{4} = (1\ 7\ 4\ 6\ 2\ 10\ 8\ 5\ 11\ 3\ 12\ 9\ 13)$$

and nally

$$^{12} = \ ^{8}\ ^{4} = (1\ 11\ 6\ 9\ 8\ 7\ 3\ 2\ 13\ 5\ 4\ 12\ 10)$$

To nd I then wrote it as a 2-cycle representation

$$= (a\ 1)(b\ 2)(c\ 3)(d\ 4) \quad (m\ 13)$$

and work through, from left to right, making sure I put the numbers back where they started. For instance $^{12}(1) = 11$, so set $a = 11$, $^{12}(2) = 13$, so $b = 13$, $^{12}(3) = 2$ so $c = 13$ (I've already made $b = 13$, and so far $2 ! \ 13$ so now I make $13 \ 3$ after, so that overall $2 ! \ 13$). Continuing, we nd

$$= (11\ 1)(13\ 2)(13\ 3)(12\ 4)(12\ 5)(9\ 6)(13\ 7)(13\ 8)(13\ 9)(11\ 10)(11\ 12)(11\ 13)$$
$$= (1\ 10\ 12\ 4\ 5\ 13\ 2\ 3\ 7\ 8\ 9\ 6\ 11)$$

Finally, this means the cards originally ordered $A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K$ become, after one shu e, $J; K; 2; Q; 4; 9; 3; 7; 8; A; 6; 10; 5$.

3. (a) Draw the right angled triangle ABC with right angle at C. Let D be the midpoint of AB, and E a point on AC such that AC ? DE. Then ADE is similar to ABC (three angles equal). Since $AD = \frac{1}{2}AB$ then $AE = \frac{1}{2}AC$ or rather $AE = EC$. Now AED is congruent to CED (two sides equal, $AE = EC$, DE common, and an included angle $\setminus AED = \setminus DEC$). Thus $\frac{1}{2}AB = AD = DC$.

   (b) From part i) we see $DB_1 = B_1C$ and $DC_1 = C_1B$. Note that CB_1A_1 is similar to CAB (two sides in ratio and an included angle). The sides are in ratio 1 : 2 so $A_1B_1 = \frac{1}{2}AB = C_1B$, and so $A_1B_1 = DC_1$. Similarly BC_1A_1 is similar to BAC, so $C_1A_1 = B_1C = B_1A_1$. Thus B_1C_1D and $B_1C_1A_1$ are congruent because they have 3 equal sides.

4. Following the hint, we must have $3m \ 1 = n$ or $3m \ 1 = 2n$, since $3m \ 1 < 3n$. So

$$3(3m \ 1) \ 1 = km; \quad k \ 2 \ Z$$
$$(9 \ k)m = 4$$
$$m = \frac{4}{9 \ k}$$
$$m = 4; 2; \text{ or } 1;$$

or

$$3\frac{3m-1}{2} - 1 = km$$
$$9m - 3 - 2 = 2km$$
$$m = \frac{5}{9-2k}$$
$$m = 5, \text{ or } 1.$$

Thus the pairs are $(1,1)$, $(1,2)$, $(2,5)$, $(4,11)$ and $(5,7)$.

5. (a) $\phi(12) = 4$, $\phi(30) = 8$

   (b) We can think of $\phi(n)$ as being the number of numbers less than n which are not a multiple of a factor of n (except the factor 1). So if p is prime, its only factors are 1 and p, so every other number is not a multiple of a factor that isn't 1, except p itself. Thus $\phi(p) = p - 1$.

   For p^2, the factors are $1, p$ and p^2, so the multiples of the factors that aren't 1 are $p, 2p, 3p, \ldots, p^2$, of which there are p. So $\phi(p^2) = p^2 - p$.

   For p^3, the factors are $1, p, p^2$ and p^3, so the multiples of the factors that aren't 1 are $p, 2p, 3p, \ldots, p^2, (p+1)p, \ldots, 2p^2, (2p+1)p, \ldots$, that is, the multiples of p^2 are contained in the multiples of p, of which there are p^2. So $\phi(p^3) = p^3 - p^2$.

   (c) Using the same method as above, the factors of pq are $1, p, q$ and pq, so the multiples of the factors that aren't 1 are $p, 2p, 3p, \ldots, qp$ (q of them) and $q, 2q, 3q, \ldots, pq$ (p of them), but we don't want to count pq twice. So $\phi(pq) = pq - q - (p - 1)$.

6. We use the fact that the medians divide $\triangle ABC$ into 2 equal area pieces, and that S is $\frac{2}{3}$ along the median from A (you can prove these by considering the areas of smaller triangles with the same heights).

   Let the median from A meet BC at P, since ST is parallel to BC triangles APC and AST are similar - 3 angles equal. Since $AS = \frac{2}{3}AP$ then the area of $\triangle AST$ is $\frac{4}{9}$ the area of $\triangle APC$ which is half the area of $\triangle ABC$ so the area of $\triangle AST$ is $\frac{2}{9}$ the area of $\triangle ABC$.

Senior Questions

1. Let $f(x) = 2x^n - nx^2 + 1$, then $f'(x) = 2nx(x^{n-2} - 1)$. So f has stationary points at $x = 0$ and $x = 1$ (since $n > 3$ and odd). Taking the second derivative $f''(x) = 2n(n-1)x^{n-2} - 2n$, so $f''(0) = -2n < 0$ and $f''(1) = 2n(n-1) - 2n = 2n(n-2) > 0$. So $x = 0$ is a local max and $x = 1$ is a local min.

   Finally $f(0) = 1 > 0$ and $f(1) = 3 - n < 0$. Since these are the only stationary points, f is monotonic between/outside of them. Since $x = 0$ is a local max, and positive there is one root for $x < 0$, which is unique since f is monotonic decreasing for $x < 0$. Since $f(0) > 0 > f(1)$ and f is monotonic between 0 and 1 there is exactly one root for $0 < x < 1$. Since $x = 1$ is a local min, $f(1) < 0$ and $f(x)$ is monotonic increasing for $x > 1$ there is exactly one root for $x > 1$. Thus, in total, there are 3 roots.

2. Take the log of both sides and the differentiate both sides with respect to x.

$$\log f(x) = x\log\left(1+\frac{1}{x}\right)$$
$$\frac{f'(x)}{f(x)} = \log\left(1+\frac{1}{x}\right) - \frac{1}{x+1}:$$

3. Draw the graph of $y = \frac{1}{t}$ for t between 1 and $1+\frac{1}{x}$ and we see that the area under the curve is larger than the area of the rectangle with base $1+\frac{1}{x} - 1$ and height $\frac{1}{1+\frac{1}{x}}$, so

$$\int_1^{1+\frac{1}{x}} \frac{1}{t}\,dt = \log\left(1+\frac{1}{x}\right) > \frac{1}{x}\frac{x}{x+1} = \frac{1}{1+x}:$$

Thus $\frac{f'(x)}{f(x)} > 0$, and since $f(x) > 0$ for all x so is $f'(x)$.