MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 19, 2014[1]

1. Let $N = 1^9 \cdot 2^8 \cdot 3^7 \cdot 4^6 \cdot 5^5 \cdot 6^4 \cdot 7^3 \cdot 8^2 \cdot 9^1$. How many perfect squares divide N?

2. Let $-10 \le a, b, c \le 10$. How many triplets, (a, b, c), satisfy

$$\frac{\frac{a}{b}}{c} = \frac{a}{\frac{b}{c}}?$$

3. Find the sum of all primes p such that $5^p + 4p^4$ is a perfect square.

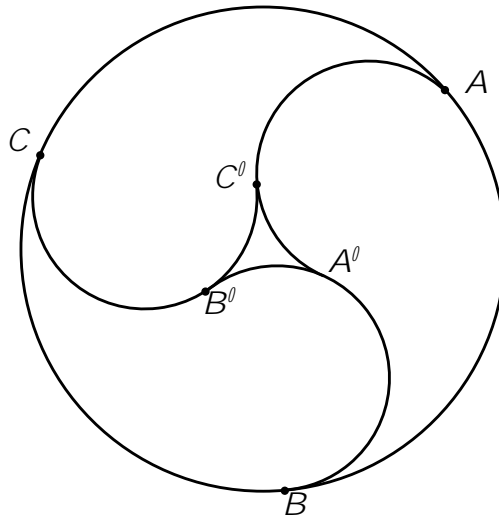4. Show that $(1 + \sqrt{5})^n + (1 - \sqrt{5})^n$ is an even integer for all positive integers n.

Figure 1: Figure for question 5

5. In the figure, ABC is a circle of radius R with 3 tear-drop shapes inside. Each of the arcs $AC'A'$, $BA'B'$ and $CB'C'$ are of circles of the same radius, r. Find the ratio of R to r and the proportional area enclosed in the centre piece $A'B'C'$.

6. Arrange 11 points in the plane so that 16 lines can be drawn, each passing through 3 points.

Senior Questions

1. Show that, for $x \in (-1; 1)$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$$

and hence show that

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots :$$

2. Using the above, how many terms are needed to approximate $\ln(2)$ correctly to 5 decimal places?