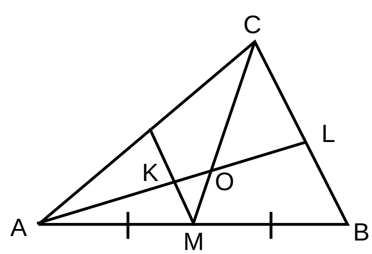
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 10, July 21, 2015[1]

1. Find the sum of all n-digits long numbers formed by $1, 2, 3, \ldots, n$. For example, if $n = 3$ then the sum of all 3-digit long numbers is $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

2. Evaluate $\sqrt[4]{2}\,\sqrt[8]{4}\,\sqrt[16]{8}\,\sqrt[32]{16}\,\sqrt[64]{32}\ldots$.

3. Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, 2, 4, 8,. . .). What is the maximal possible number of different integers on the blackboard?

4. Bob is building two roads to connect the points A and B. For any real number x, the two roads must have a length ratio of $\sqrt{(x+4)^2+4}$ to $\sqrt{(x-4)^2+16}$. Bob picks x then claims his design gives the shortest combine length of the two roads, what must this combine length be?



5. For a triangle $\triangle ABC$, M is the midpoint of the side AB and L is some point along the side BC. Let O be the point of intersection between the lines LA and MC, and let K be the point of intersection between LA and the line passing through M, parallel to BC; see above

   (a) Show that the triangles $\triangle KMO$ and $\triangle OLC$ are similar.

   (b) Suppose the length LA is twice as long as MC, and $\angle OLC = 45^\circ$. Prove LA is perpendicular to MC.

6. Consider the polynomial $p(x) = x^4 + 37x^3 + 71x^2 + 18x + 3$. If a, b, c and d are roots of $p(x)$, find a polynomial whose roots are $\frac{abc}{d}, \frac{acd}{b}, \frac{abd}{c}$ and $\frac{bcd}{a}$.

[1] Some problems from *Tournament of Towns in Toronto*.

Senior Questions

The following questions concerns the irrationality of . Recall that a number is irrational if it can not be written as $\frac{a}{b}$, where a and b are positive integers. We will study a function de ned by

$$f(x) = \frac{x^n(a \quad bx)^n}{n!};$$

where n is some positive integer.

1. Let $f^{(k)}(x)$ denote the k^{th} derivative of f, where $k = 0; 1; 2; \ldots$. Show that for each k
   (a) $f^{(k)}(0)$ is an integer.
   (b) $f^{(k)}(0) = (\quad 1)^k f^{(k)}(\quad)$.
2. Let $G(x) = f(x) \quad f^{(2)}(x) \quad f^{(4)}(x) + f^{(6)}(x) \quad \ldots + (\quad 1)^n f^{(2n+2)}(x)$.
   (a) Show that $f(x) = G(x) + G^{(2)}(x)$.
   (b) By considering the function $G^{(1)}\sin(x) \quad G(x)\cos(x)$ and the result of part (a), show that $\int_0 f(x)\sin(x)\,dx = G(0) \quad G(\quad)$.

3J 0