

MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, August 25, 2015¹

1. Write $2^{32} + 2^{17} + 1 = (2^{16} + 1)^2$, and observe that $2^{16} + 1$ is the largest Fermat prime number.
2. First we show that the 5^{th} power of any integer have the same last digit as the original number. For a n -digit long integer x , we can write it as $x = a_1 + 10a_2 + \dots + 10^{n-1}a_n$, where $a_1; a_2; \dots; a_n$ are nonnegative integers less than 10. Then

$$x^2 = a_1^2 + 10a_1a_2 + 10^2a_2^2 + \dots$$

in words, the last digit of x^2 will be the same as the last digit of a_1^2 . Repeating this calculation, we can show that the last digit of x

$$\frac{x^2}{a^2} = \frac{10^n + 1}{a};$$

Since $10^n + 1$ is a $n + 1$ digits long number, the fraction on the RHS of the last equation must be greater than 0 and less than 10. Therefore, the integer $1 < k < 9$. Since $ak = 10^n + 1$, k must be odd. k can not be 1, otherwise a will be $n + 1$ digits long. k can not be 3; 5 or 9, because $10^n + 1$ is never divisible by those numbers. Thus the only possibility is 7.

5. (a) If we draw a horizontal line across any one of the 1×1 grid squares (we think of the dimension as *vertical* / *horizontal*), then no matter how the 10×12 paper is folded along the grid lines, this horizontal line will still be horizontal. Similarly, any vertical line will be preserved under the action of folding.

¹Some problems from UNSW's publication *Parabola*, and the Tournament of Towns in Toronto.

Thus, if we were to cut horizontally across the thick 1×1 square folded paper, then we are cutting horizontally across each of 1×1 grid squares of the unfold paper. This means we will produce $10 + 1$ strips of papers. Similarly, if we cut vertically then we will produce $12 + 1$ strips.

- (b) Label the first row along the grid points (i.e the corner of the 1×1 squares) with A 's and B 's in a alternating fashion. Then label the second row with C 's and D 's in an alternating fashion. Repeat the AB labelling on the third row, and CD on the fourth row etc, in an alternation fashion until each grid point has been labelled. Then no matter how the 10×12 paper is folded, the $A; B; C$ and D corners will always be folded onto itself.

Thus, if we were to cut the corner labelled by A of the thick 1×1 square folded paper, then we would be cutting out each of the grid labelled with an A in the original unfold 10×12 paper. Since there are 5×6 grid points labelled with an A , it follows that there are $30 + 1$ separated pieces due to this cut. We can use a similar argument to work the number of components we get by cutting the other corners of the folded 1×1 paper.

6. Let $a = 10^b$, then we can rewrite the inequality $10 < a^x < 100$ as $1 < bx < 2$. Similarly, if $100 < a^x < 1000$, then $2 < bx < 3$. Support 6alw with an

