# MATHEMATICS ENRICHMENT CLUB.

## Solution Sheet 17, September 8, 2015[1]

1. If the number is made from $a \neq 1$, then $aaa\ldots$ is divisible by $a$ and thus not prime.

   Suppose the number has a non-prime number of digits, then we can factor the number of digit this number has into $q \times p$ where $q$ and $p$ are integers. Hence, we we can "split" the number up into $q$ blocks of $p$-length digits; i.e

   $$\underbrace{111\ldots1}_{q \times q \text{ lots of 1's}} = \underbrace{\underbrace{111\ldots1}_{p \text{ lots of 1's}}\ \underbrace{111\ldots1}_{p \text{ lots of 1's}}\ \cdots\ \underbrace{111\ldots1}_{p \text{ lots of 1's}}}_{q \text{ lots of blocks}}.$$

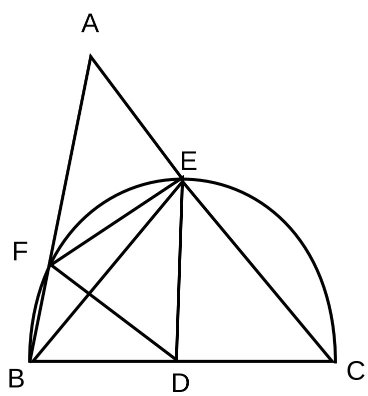
   The RHS of the number above is divisible by $\underbrace{111\ldots1}_{p \text{ lots of 1's}}$.

2. For a number to be divisible by 9, the sum of its digits must also be divisible by 9. If each digit of the number is even then so is the sum of its digits. So we start with smallest sum of digits that is divisible by 9 and even; this number is 18. It is easy to check that the number has at last 3-digits, so 288 is the smallest possible solution.

3. Since the RHS of $(m - 8)(m - 10) = 2^n$ is a power of 2, both $m - 8$ and $m - 10$ must be

(b) We show that we can not eat all nuts if at any moment the total number of nuts the children have is 3, then we are done because only one nut will be consumed at any time and we started with more than 3. Due to the argument in part (*a*), one child will have zero nuts, so the number of nuts each child have is $0, 2, 1$. Also, observe that the child with the highest number of nuts will be the next to do the dividing, so after another iteration, the number of nuts each child will have is $1, 0, 2$; this forms an endless loop.

5. $\triangle ABC$ can be equilateral, but we can construct an example where it is not: Let $\triangle DEF$ be an equilateral triangle. Construct a semicircle with centre $D$ and radius $DE$. The diameter $BC$ of the semicircle is perpendicular to $DE$, with $F$ closer to $B$ than to $C$. Since $DF = DE$, $F$ also lies on this semicircle. Extend $BF$ and $CE$ to meet at $A$. Since $\angle BEC = 90^\circ = \angle BFC$, $BE$ and $CF$ are indeed altitudes of triangle $ABC$. Since $A$ lies on the extension of $CE$ and $DE$ is the perpendicular bisector of $BC$, $AB < AC$. Hence $ABC$ is not equilateral.




6. If we compare the sum $1 + 2 + 3 + \dots + 14 = 7 \times 15$ to $16 + 17 + 18 + \dots 29 = 7 \times 45$, we see that latter is three times greater. Thus, all numbers $< 15$ must be place below the main diagonal, and all numbers $> 15$ above it. Hence, the remaining 29 lots of 15's must be place at the diagonal, which implies the central block is 15.

## Senior Questions

1.

$$T_n \left( \frac{1}{T_1} + \frac{1}{T_2} + \dots + \frac{1}{T_n} \right) = n^2$$

or

$$\frac{1}{T_1} + \frac{1}{T_2} + \dots + \frac{1}{T_n} = \frac{2n}{2n+1}.$$

This can be proven using induction. The inductive step depends on

$$\frac{2n}{n+1} + \frac{a}{(n+1)(n+2)/2} = \frac{2(n+1)}{(n+1)+1}.$$

2. Let $(0; t)$ be the point of intersection of $AB$ and $CD$. Then the equation of the line $AB$ is given by $\frac{y}{x-t} = \frac{a^2-b^2}{a-b} = a + b$. That $A$ lies on this line means that $\frac{a^2}{a-t} = a + b$. We have $a^2 = a^2 ab \quad t(a + b)$ so that $t = \frac{ab}{a+b}$. Similarly, $t = \frac{cd}{c+d}$. Eliminating