

**MATHEMATICS ENRICHMENT CLUB.**  
Solutions Sheet 8, June 16, 2015<sup>1</sup>

1. We have

$$\begin{aligned}x^2 - y^2 &= 1999 \\(x - y)(x + y) &= 1999;\end{aligned}$$

and since 1999 is prime, either  $(x - y) = 1$ ,  $(x + y) = 1999$  or  $(x + y) = -1$ ,  $(x - y) = -1999$ ; there is a total of four integral solutions to both cases.

2. There are two solutions to this problem: One uses the perimeter of  $\triangle ABC$ , the other uses the area of  $\triangle ABC$ .

(a) Hint: Let the point of intersection between the circle and the side  $AB$  be  $P$ . Then the radius of the inscribed circle is  $|AP|$ , and the line  $BP$  is tangent to the circle at the point  $P$ .

(b) Hint: Let  $M$  be the middle of the inscribed circle. Then the triangles  $\triangle ABM$ ,  $\triangle BCM$  and  $\triangle CAM$  all have height equal to the radius of the inscribed circle.

3. A neat trick is to express  $N$  as

$$\underbrace{333 \dots 333}_{61 \text{ 3's}} = \frac{3}{9} \underbrace{999 \dots 999}_{61 \text{ 9's}} = \frac{3}{9}(10^{61} - 1):$$

Similarly,  $M = \underbrace{666 \dots 666}_{62 \text{ 6's}} = \frac{6}{9}(10^{62} - 1)$ . Now

$$\begin{aligned}NM &= \frac{2}{9}(10^{61} - 1)(10^{62} - 1) \\&= \frac{2}{9}(10^{61} - 1) \cdot 10^{62} \cdot \frac{2}{9}(10^{61} - 1) \\&= \underbrace{222 \dots 222}_{61 \text{ 2's}} \underbrace{000 \dots 000}_{62 \text{ 0's}} \underbrace{222 \dots 222}_{61 \text{ 2's}} \\&= \underbrace{222 \dots 222}_{60 \text{ 2's}} 19 \underbrace{777 \dots 777}_{60 \text{ 7's}} 8:\end{aligned}$$

<sup>1</sup>Some problems from UNSW's publication Parabola

It is easy to compute the sum of digits on the last line of the above equation; it is 558.

4. First we note that for positive integers  $m; n; k$  and  $r$ , if  $a = mk + r$ , then  $a^x = nk + r^x$  (you may want to show this is true). Now, since  $a = x$

then by using the assumption that  $x < y$ , we have

$$d \frac{x-y}{2} + 1 - \frac{y}{x} = \frac{x}{2}$$

$$d \frac{x-y}{2} + 1 - \frac{x}{y} = \frac{y}{2}$$

The last system of inequality does not hold because  $x < y$ , so we have a contradiction to the advertisement's claim.

### Senior Questions

1. Since  $c > 0$ ,  $x^2 + 1 = x^2 + \frac{1}{2} + 2$  2. Similarly,  $x^2 + 1 = x^2 + \frac{1}{2} + 2$ . Therefore, if  $r_1$  and  $r_2$  are the roots of  $f$  (assuming  $r_1 > r_2$  wlog). Then  $r_1 > 2$  and  $r_2 < 0$ , so that  $r_1 r_2 = c - 3 < 0$ , which implies  $c < 3$ .

To get the lower bound on  $c$ , we use the quadratic formula  $r_1 = \frac{(c+1) + \sqrt{(c+1)^2 - 4(c-3)}}{2}$ . Solving gives  $c \geq 3$ .

2. Lets start by looking at the extreme case  $BX = XC; CY = YA; AZ = ZB$ ; as shown below. By the Midpoint Theorem, the line  $BC$  is parallel to  $ZY$ , and the line  $AC$  is parallel to  $ZX$

Since the RHS of the above equation is rational,  $\sqrt[p]{c}$  must be rational. Write  $\sqrt[p]{c} = \frac{x}{y}$ , where  $x$  and  $y$  are integers with greatest common multiplier one. Then  $c = \frac{x^p}{y^p}$ , and greatest common multiplier between  $x^p$  and  $y^p$  is one. Since  $c$  is an integer,  $x^p$  must