MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 4, May 19, 2015[1]

1. Let a and b be positive integers such that $2^a - 2^b = 2016$. Find the value of $a + b$.

2. Let $ABCD$ be a square, with M and N the mid points of the sides BC and AD respectively. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

3. Find the smallest positive integer n such that $\frac{1}{3}n$ is a perfect cube, $\frac{1}{5}n$ a perfect fifth power and $\frac{1}{7}n$ a perfect seventh power.

4. Simplify
$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1}.$$

5. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$. Hint: Interpret this problem geometrically.

6. In how many ways can we choose n integers $x_1, x_2, \ldots, x_n$ such that each is $0, 1$ or 2 and their sum is even?

Senior Questions

1. Given that a, b, and c are positive integers, solve

   (a) $a!b! = a! + b!$

   (b) $a!b! = a! + b! + 2^c$

   (c) $a!b! = a! + b! + c!$

2. (a) Prove that for $n \geq 3$, $(n+1)! > (n-2)(1! + 2! + \ldots + n!)$.

   (b) Use part (a) or otherwise, show that for $n \geq 3$, $(n+1)!$ is not divisible by $1! + 2! + \ldots + n!$.