

MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 14, August 22, 2016

1. By the property of the perpendicular from the midpoint $NA = NB$, thus triangle $\triangle ANB$ is isosceles. Angle A is equal to 60° , this means that $\triangle ANB$ is equilateral and

4. By Pythagoras, the equation $p^2 + q^2 = 1$ implies that p and q are the sides of a right-angled triangle. Hence, there is an angle θ such that

$$p = \cos \theta \quad \text{and} \quad q = \sin \theta :$$

Similarly, the equation $r^2 + s^2 = 1$ implies that there is an angle ϕ such that

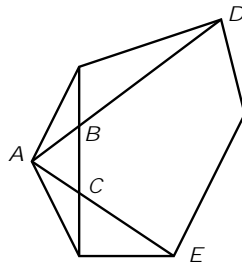
$$r = \cos \phi \quad \text{and} \quad s = \sin \phi :$$

Therefore

$$(pr + qs)^2 = (\cos \theta \cos \phi + \sin \theta \sin \phi)^2 = \cos^2(\theta - \phi) \leq 1 :$$

5. We can classify the triangles in the n -gon by the number of vertices of the triangle that are also vertices of the n -gon. There are four possible types of triangles $t_0; t_1; t_2; t_3$, where the subindex denotes the number of common vertices.

We claim that there are no type t_1 or t_0 triangles. For the t_1 triangles, let A be the only vertex of the triangle that is also a vertex of the n -gon, and $D; E$ two other vertices of the n -gon. Then there must be at least two diagonals AD, AE that are incident on A , with the other two vertices of the triangle B and C points intersection between AD, AE and another diagonal of the n -gon; see below. Then $BDEC$ can not be formed into triangles, since there can not be another diagonal that passes through B or C . Hence, there are no t_1 triangles. By similar arguments, we can show that there are no t_0 triangles.



Now suppose we partition the n -gon into as many t_3 triangles as we can. Then for any two adjacent t_3 triangles, we can always partition it into two t_2 triangles using another diagonal of the n -gon. Since the number of t_3 triangles in the n -gon is $n - 2$, if n is even, then the maximum number of triangles we can make is $2(n - 2)$ t_2 triangles. If n

We have

$$\begin{aligned}\frac{1}{9} &= 0.1111111111 \dots \\ \frac{1}{99} &= 0.0101010100 \dots \\ \frac{1}{999} &= 0.0010010010 \dots \\ \frac{1}{9999} &= 0.0001000100 \dots\end{aligned}$$

and so on. Let s_i be the sum of the digits on the i^{th} decimal place. Then, the first number $1=9$ has a 1 in every place, the next number $1=99$ has 1 every second place, the next number $1=999$ every third, and so on. The sum s_i is equal to the number of factors in i . Since 17 is prime, the only factors of 17 are 1 and 17. Hence $s_{17} = 2$. However, we must also account for the possible "carry" over from summing the digits $s_{18}; s_{19}; \dots$. The total in the 18th place, including all carries, is

$$t_{18} = s_{18} + \frac{s_{19}}{10} + \frac{s_{20}}{100} + \frac{s_{21}}{1000} + \dots$$

We can make the approximations

$$s_{19} < 19; \quad s_{20} < 20 < 2 \cdot 19; \quad s_{21} < 21 < 2^2 \cdot 19 \dots$$

Hence,

$$\begin{aligned}t_{18} &< 5 + \frac{19}{10} + \frac{2 \cdot 19}{100} + \frac{2^2 \cdot 19}{1000} + \dots \\ &= 5 + \frac{19}{10} \left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots \right) \\ &= 5 + \frac{19}{10} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{59}{8} < 10.\end{aligned}$$

Hence, there is no carry over from t_{18} .

Senior Questions

- (a) The answer is yes. Andy can call the cards out in order starting with the Ace of Spades, two of Spades down to the King of Spades, followed by the Hearts, the Diamonds and the Clubs. We refer to this as one cycle. In each cycle, each card can move at most once since it is called exactly once, and at least one card must move. Andy then makes another 51 cycles of calls. We claim that all moves are in the same direction, either all clockwise or all counter-clockwise. This is clear within each cycle. Consider the card X which is the last to move in a cycle, and let Y be the other card adjacent to the empty spot. Since Y does not move after X

and this is not possible since Andy makes only 52 cycles of calls. If it is to be in its initial spot, it must not have moved at all. However, this is also impossible as otherwise at most 1 move could have been made, but in 52 cycles, at least 52 moves have been made. Therefore, after 52 cycles of calls, every card is in a spot different from its initial one

- (b) The answer is no. Construct a graph where each of the vertices represents one of the $52!$ permutations of the cards, with the first and the last adjacent to the empty spot. Two vertices are joined by an edge if and only if a call by Andy changes the two permutations to each other. Label the edge with the card called by Andy. In this graph, each vertex has degree 2, and the graph is a union of disjoint cycles. Consider the cycle containing the vertex representing the initial permutation. For each vertex, let a person starts there. Whenever Andy makes a call, the person moves along an edge labelled with that card to an adjacent vertex if possible, and stays put otherwise. We call a vertex safe if and only if in the permutation it represents, the Queen of Spades is not adjacent to the empty spot. By shifting each card clockwise into the empty spot in turns, we will arrive at permutations represented by safe vertices as well as permutations represented by unsafe vertices. Note that after each call, there is still one person on each vertex. Thus no matter what sequence of calls Andy may employ, he cannot get everyone to a safe vertex. It follows that there is an initial permutation for which Andy's sequence will leave the Queen of Spades adjacent to the empty spot.
2. Draw a chord through A and P , intersecting the circle again at M , and a chord through B and P , intersecting the circle again at N . Let X be the intersection of lines AN and BM . Since the angle in a semicircle is right angle, AM and BN are altitudes of $\triangle ABX$. But the three altitudes of a triangle are concurrent.

Hence $x^n + y^n$ is not prime, since it contains the factor $x + y \neq 2$.

Suppose n is even, then by assumption $n = mk$, where k is odd. Hence

$$\begin{aligned}
 & x^n + y^n \\
 &= x^m x^k + y^m y^k \\
 &= (x^m)^k + (y^m)^k \\
 &= (x^m + y^m)((x^m)^{n-1} - (x^m)^{n-2}y^m + (x^m)^{n-3}(y^m)^2 - \dots + (y^m)^2(x^m)^{n-3} - y^m(x^m)^{n-2} + (y^m)^{n-1}).
 \end{aligned}$$

Hence $x^n + y^n$ is not prime, since it contains the factor $x^m + y^m \neq 2$.