MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 15, August 29, 2016

1. Since $1 + 2 = 3$ is prime, we know that n 2. We show that $n = 2$. Let $f(x)$ be the sum of n consecutive positive integers starting from x. Then

$$f(x) = x$$

even. Hence, the last digit of x_1 will change by at least 2 and at most 8 and the second last digit of x_1 will change by at most 1.

Now since x_1 is odd, the last digit of x_1 is odd. Since the largest digit of x_1 is even, the last digit of x_1 can not be the largest digit of x_1.ofx_2

board. To calculate the number m_0 of unique paths the king can take, we can think of the king picking from 10 possible moves, without caring about the order of the 5 individual up moves, or the 5 individual right moves he makes; that is

$$m_0 = \frac{10!}{5! \cdot 5!},$$

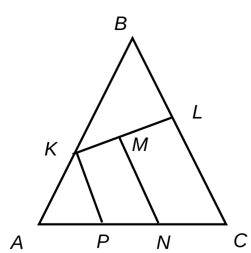
where $10! = 10 \times 9 \times 8 \ldots$ (the number of ways to order 10 objects without replacement).

Now, suppose the king takes one diagonal. Then the king must take 4 moves right, 4 moves up, and 1 mover diagonally to get to the top right hand corner of the chess board. The number m_1 of unique paths the king can take in this fashion is

$$m_1 = \frac{10!}{4! \cdot 4! \cdot 1!}.$$

Since the king can make up to 5 diagonal moves, repeating the above calculations for m_2, m_3, m_4 and m_5 then adding yields 1683 possible ways the king can move to the top right hand corner.

5. Draw a straight line KP parallel to LC, where P is a point on AC. Then $KLCP$ is a trapezoid, hence its mid-line $MN = \frac{1}{2}(KP + LC) = \frac{1}{2}(AK + LC) = \frac{1}{2}KL = KM = ML$. Therefore KL is a diameter of a circle passing through K, N, L. Thus, $\angle KNL = 90^\circ$.

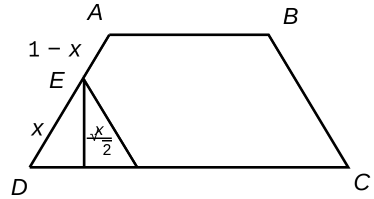


6. It is possible to show that $ABCD$ is isosceles, with base angles of 45°. Let $DE = x$, then $AE = 1 - x$, and we obtain the equation

$$1 - x \quad x/\sqrt{2}.$$

This yields

$$x \quad 2 - \sqrt{2}.$$



Senior Questions

1. Let x_1 and x_2 be integers, such that (x_1, y_1) and (x_2, y_2) are two points on the given polynomial. Then

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots,$$

and

$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots,$$

for some integers $a_0, a_1, a_2, \ldots$. Hence

$$y_1 - y_2 = a_1(x_1 - x_2) + a_2(x_1^2 - x_2^2) + \ldots. \qquad (1)$$

Since $x_1^k - x_2^k$ is always divisible by $x_1 - x_2$ for all integers k, the RHS of (1) is divisible by $x_1 - x_2$. Thus, $y_1 - y_2 = n(x_1 - x_2)$ for some integer n.

Now the distance d between (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + n^2)}, \qquad (2)$$

since $y_1 - y_2 = n(x_1 - x_2)$ for some integer n. We show that if d is an integer, then the gradient g given by

$$g = \frac{y_1 - y_2}{x_1 - x_2},$$

must be 0. Thus completing the proof.

If d is an integer, then by (1), the expression $1 + n^2$ must be square of some integer, which implies $n = 0$. But if $n = 0$, then $y_1 - y_2 = n(x_1 - x_2) = 0$. Therefore, $g = 0$.

2. Since $(x - 1)(x - 2)$ is a quadratic, the remainder of the x^{2016} divided by $(x - 1)(x - 2)$ must be of the form $ax + b$, for some integers a, b. Hence

$$x^{2016} = (x - 1)(x - 2)f(x) + ax + b, \qquad (3)$$

where $f(x)$ is a polynomial of degree 2014. We can