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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 17, September 12, 2016

1. We can factor the LHS of this equation as follows

   $$x^2 + xy - y - 1 = x^2 - 1 + y(x - 1) = (x + 1)(x - 1) + y(x - 1) = (x + y + 1)(x - 1).$$

   Hence,
   $$(x + y + 1)(x - 1) = 2016 = 2^5 \cdot 3^2 \cdot 7.$$

   Now since $x + y + 1 > x - 1$, one possible solution satisfies $2^4 \cdot 3^2 \cdot 7 = x + y + 1$ and $x - 1 = 2$; that is, $x = 3, y = 1004$ is a solution. Other solutions pair are found similarly.

2. Suppose c on them, and $c = -a$. However, if we add 4 cards with the number a on them, and 1 card with the number b on it, then it is not possible to add another 5 cards to obtain a sum of 0, unless $a = c = 0$. Hence, this contradicts $a, b, c \neq 0$.

3. Let x be the distance between the two legal firms, v_a, v_b the walking speeds of Albert and Betty respectively, and t_1, t_2 the time elapsed when Albert and Betty pass each the first and second time respectively. Then

   $$t_1 v_a = A \quad \text{and} \quad t_1 v_b = x - A.$$

   Thus,

   $$\frac{A}{v_a} = \frac{x - A}{v_b}$$
   $$\Rightarrow \frac{v_a}{v_b} = \frac{x - A}{A}$$

   Similarly, on the return trip

   $$t_2 v_a = x + B \quad \text{and} \quad t_2 v_b = 2x - B$$

Thus,

$$\frac{v_a}{v_b} = \frac{2x - B}{x + B}.$$

Therefore

$$\frac{x - A}{A} = \frac{2x - B}{x + B},$$

so that $2Ax - AB = x^2 - Ax + Bx + AB$; that is, $x(3A - B - x) = 0$. Hence, $x = 0$ or $x = 3A - B$. The distance they walked is thus $2x = 6A - 2B$.

Also, $\frac{v_a}{v_b} = \frac{A}{2A-B}$. So if $A < B$, then $v_a > v_b$.

4. Not true. Consider the sequence $a_{2k-1} = k^2$, $a_{2k} = k(k+1)$ for all $k \geq 1$. The sequence a_k alternates between geometric and arithmetic sums.

5. Let $ABCD$ be a quadrilateral such that the diagonals are at right angles at the point X. Pythagoras gives

$$|AB|^2 = |AX|^2 + |BX|^2,$$
$$|BC|^2 = |BX|^2 + |CX|^2,$$
$$|CD|^2 = |CX|^2 + |DX|^2,$$
$$|DA|^2 = |DX|^2 + |AX|^2.$$

Hence $|AB|^2 + |CD|^2 = |BC|^2 + |DA|^2$ or $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$.

Suppose we deform the corners of the quadrilateral, then $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$ still holds, since the size of the edges had not change. We show that in the deformed quadrilateral $AC \perp BD$.

In the triangle ABC drop a perpendicular BP from B to AC. Then $|AB|^2 - |BC|^2 = |AP|^2 + |BP|^2 - (|PC|^2 + |PB|^2) = |AP|^2 - |PC|^2$. Similarly in triangle ADC drop perpendicular DQ to AC, then $|AD|^2 - |DC|^2 = |AQ|^2 - |QC|^2$. Since $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$, we have $|AP|^2 - |PC|^2 = |AQ|^2 - |QC|^2$, so $|AP|^2 + |QC|^2 = |AQ|^2 + |PC|^2$. This can only be true if $P = Q$. Hence BP and $DQ = DP$ are both perpendicular to AC, and so $BD \perp AC$.

6. Let O be the circumcentre of the triangle MAN. Then $\angle MON = 60^\circ$, and $ON = OM$. Therefore, MON is an quadrilateral triangle, which implies $\angle ONM = \angle OMN$. Now C is an external point of the circle centered at O with radius $|ON|$, and is a common point of the lines NC and MC. Thus NC and MC are tangent to a circle, which implies $\angle CNO = \angle CMO = 90^\circ$. Therefore $ONCM$ is a cyclic quadrilateral. Since ONC is cyclic, $\angle BCO = \angle OMN = \angle ONM = \angle DCO$. Hence, the line CO bisects $\angle BCD$, and O lays on the diagonal AC of $ABCD$.

Senior Questions

1. .

2. Consider the removal of two consecutive digits 0 from an arbitrary binary number (if any exist), we claim that the action of removing 00 reduces the number of odd and even pairs by the same amount. Thus this operation can be applied to reduce the complexity of this problem. Fix a digit 1 from the arbitrary binary number, if this digit 1 is after 00, then removing 00 does not a ect the number of even or odd pairs linked with this digit 1. On the other hand, if this 1 comes before 00, then there are two cases to consider: either one of the digit from 00 is pair with this 1 or it is not. If the rst 0 forms an even pair with this digit 1, then the second 0 must forms an odd pair with the same digit 1, and visa versa. Thus, the number of odd and even pairs