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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 9, June 27, 2016

1. Yes. Let us work backward starting from $A+1$ and applying the two allowed operations multiple times to get to A. Put the digit 1 on the left of $A+1$, to obtain a new number B_1. Then put the digit 1 on the left of B_1 to obtain B_2

Expanding the RHS of the above cubic equation and equating like powers of x gives $a + b + c = 0$ and $ab + ab + bc = 2$ and $abc = 1$. Therefore,

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc) = -4.$$

Hence, using the fact that $x^3 = 1 - 2x$ for $x = a, b, c$, we have

$$\begin{aligned} a^3 + a^2 + a + b^3 + b^2 + b + c^3 + c^2 + c &= a^2 + b^2 + c^2 - a - b - c + 3 \\ &= -4 + 0 + 3 = -1. \end{aligned}$$

4.

Senior Questions

1. Here p and q are prime. We have

$$\begin{aligned} p! + 1 &= (2p + 1)^2 \\ &= 4p^2 + 4p + 1 \\ p! &= 4p^2 + 4p \\ (p-1)! &= 4p + 4 \\ (p-1)! - p + 1 &= 3p + 5. \end{aligned}$$

   Hence

$$\frac{(p-1)! - (p-1)}{p} = \frac{3p+5}{p} = 3 + \frac{5}{p}. \qquad (2)$$

   Since p is prime, $(p-1)! - (p-1)$ is divisible by p. Thus the LHS of (2) is an integer. In particular, $3 + 5/p$ is an integer, so that $p = 5$. The unknown q is found similarly.

2. The substitution $2x = \sec u$ may help.

3. Since CE is perpendicular to AB, the triangles $\triangle AEC$ and $\triangle CEB$ are similar. Hence,

   (a) $\angle EAC = \angle ECB$, thus $\angle LAE = \angle MCE$.

   (b) $\frac{CB}{CE} = \frac{AC}{AE}$, thus $\frac{CM}{CE} = \frac{AL}{AE}$.

   It follows that $\triangle LAE$ is similar to $\triangle MCE$. Therefore, $\angle ELA = \angle EMC$, which implies the quadrilateral $LAEM$ is cyclic. Hence, $\angle LEM = \angle LAC = 90^\circ$.