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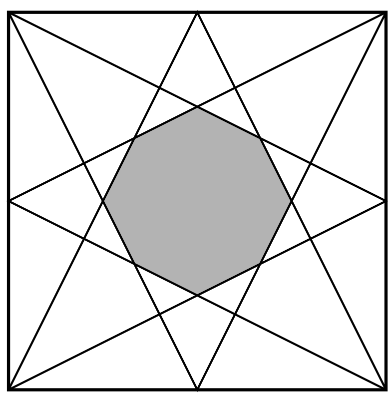
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 21, 2017

1. Given that x and y are integers, find all solutions to
$$3x^2 - 8xy + 4y^2 = -12$$

2. Write the quartic $x^4 + 4$ as the product of two quadratics. What about $x^4 + 1$?

3. Find all positive integers x, y and z such that
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{8}.$$
(Hint: Suppose $x \le y \le z$ and hence find the possible values of x.)

4. An octagon is created by joining the vertices and midpoints of the sides of a unit square as shown below.



Calculate the area of the octagon.

5. In how many ways is it possible to write 1000 as a sum of consecutive odd integers?

6. Let n be an integer greater than 1. The tau-function, $\tau(n)$ is defined as the number of divisors of n (including n itself). For example, the divisors of 6 are 1, 2, 3 and 6, so
$$\tau(6) = 4.$$
   (a) Evaluate $\tau(7)$,

Senior Questions

1. Find the sum

$$S = \frac{1}{1\times 4} + \frac{1}{4\times 7} + \cdots + \frac{1}{(3n-2)(3n+1)}$$

2. Let $I = \int \sec\theta \, d\theta$.

   In this question, we will evaluate I in two different ways.

   (a) METHOD I: Show that

   $$\sec\theta = \frac{\cos\theta}{1-\sin^2\theta}.$$

   Hence evaluate I.

   (b) METHOD II: Show that if $f(\theta) = \sec\theta + \tan\theta$, then

   $$\frac{f'(\theta)}{f(\theta)} = \frac{\sec\theta(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)}.$$

   Hence evaluate I.

   (c) Reconcile the results of Method I and Method II.

3. Let n be an integer greater than 1. The sigma-function, $\sigma(n)$ is defined as the sum of the divisors of n (including n itself). For example, the divisors of 6 are 1, 2, 3 and 6, so

$$\sigma(6) = 1 + 2 + 3 + 6 = 12.$$

   Find a formula for $\sigma(n)$.