MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 12, July 24, 2017

1. The two conditions can be written as

$$\text{(a)} \sum_{d|n} d = \sum_{t|m} t = A; \qquad \text{(b)} \sum_{d|n} \frac{1}{d} = \sum_{t|m} \frac{1}{t} = B;$$

for two positive values A and B (every positive number n has at least two divisors: 1 and n).

Note that if $d|n$ then $\frac{n}{d}$ is also a divisor of n, so in particular

$$nB = n\left(\sum_{d|n} \frac{1}{d}\right) = \sum_{d|n} \frac{n}{d} = \sum_{d|n} d = A$$

and similarly for m

$$mB = m\left(\sum_{t|m} \frac{1}{t}\right) = \sum_{t|m} t = A.$$

Putting everything together we see that

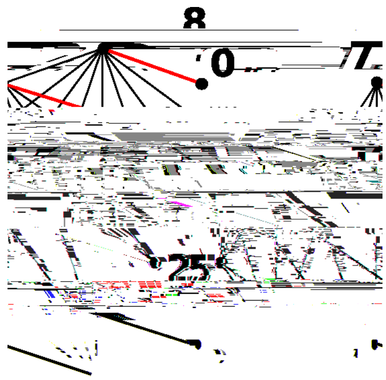
$$n = m = \frac{A}{B}.$$

2. We have to prove that no odd number has a square on it's back and squares have an even number on it's back, so we have to flip cards A and D.

3. There were 10 people in the dinner. Nine of them (all but the matematician) shook hands with different number of people: but, since none of them shook hands with their partners or with themselves, they shook at most 8 hands.

That means that among those 9 guests there has to be one person who shook 8 hands, another shook 7, ..., another shook 1 and someone did not shake hands with any of the guests.

The following diagram express the only possible outcome (black edges represent a handshake, red edges represent the couples and the numbers represent the number of people that each person shook hands with).

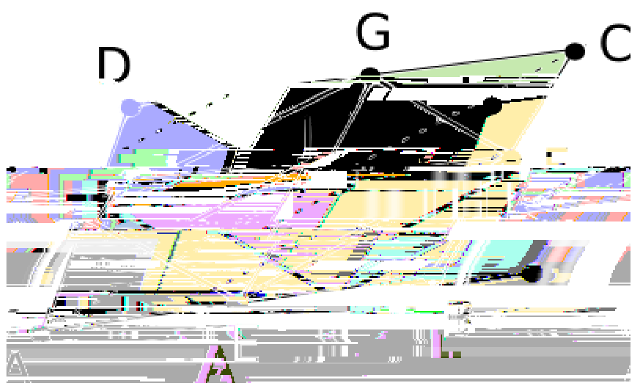


The mathematician and its partner both shook hands with four other people.

4. Let us denote the mid points (like in the picture)

$$E = \frac{A + B}{2}; \quad ; F = \frac{B + C}{2}; \quad ; G = \frac{C + D}{2}; \quad H = \frac{D + A}{2}$$

and in addition let $J = \frac{A+C}{2}$ denote the midpoint of the diagonal AC.



There are two triangles of each color. Each pair of triangles of the same color are congruent to each other, with one of the pair inside parallelogram $EFGH$ and the other outside it. By dissection, $EFGH$ must have exactly half the area of the full quadrilateral $ABCD$.

5. It is easy to see that for any integer value of p either p, $p+2$ or $p+4$ must be divisible by 3, so that means that the triplet $(3; 5; 7)$ is the only possible one.

   To see this, it suffices to consider the cases $p = 3k + 1$ or $p = 3k + 2$ (since otherwise p is already divisible by 3).

Senior Questions

1.

is either an integer (if m was a lower power of two) or it is a rational number with odd denominator. Then clearly

$$2^{k-1}H_n = 2^{k-1} + 2^{k-2} + \frac{2^{k-1}}{3} + 2^{k-3} + \cdots + \frac{1}{2} + \cdots + \frac{2^{k-1}}{n} = \frac{1}{2} + \frac{a}{b}$$

where b is an odd number. Therefore, we can write

$$\frac{1}{2} = \frac{b2^{k-1}H_n - a}{b}$$