[redacted]

# MATHEMATICS ENRICHMENT CLUB.
## Solution Sheet 13, 27 August, 2018

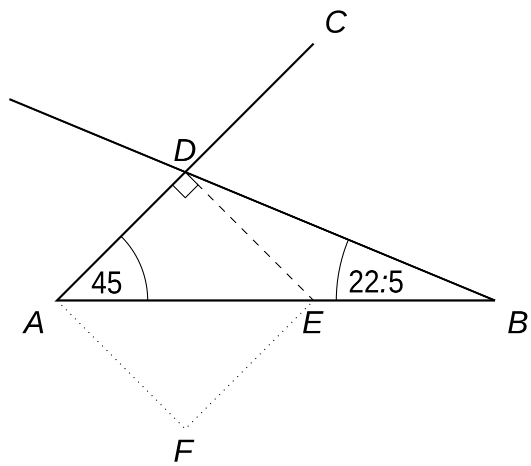
1. It is not possible. The sum of the digit sums of the numbers on the left-hand side of the equation is $1 + 2 + 3 + \dots + 9 = 45$, which is a multiple of nine, but 100 is not.

2. (a) We are choosing three objects from a possible 8, and order is not important, so there are ${}^{8}C_{3} = 56$ triangles.

   (b) Most of the triangles in the cube are right-angled. However, we can create an equilateral triangle by joining together three diagonals of the faces of the cube, e.g. *4* where $\lfloor x \rfloor$ is the largest integer smaller than $x$. Thus

$$\text{Frobenius}(5, 9, 13) = \left\lfloor \frac{5-2}{2} \right\rfloor + 1 \times 5 + (4-1)(5-1) - 1 = 2 \times 5 + 3 \times 4 - 1 = 21.$$

4. Construct another line, $AC$ at $45^\circ$ to $AB$ (by bisecting a perpendicular to $AB$ at $A$). Construct a third line at $22.5^\circ$ to $AB$ at (B) (by bisecting a perpendicular twice). Produce this line until it intersects with $AC$ at $D$. Then $AD$ is the length of the side of the square.




Proof:
Drop a perpendicular from $D$ to $E$ on $AB$ (shown with the dashed line in the diagram). Then $\angle AED = 45^\circ$, and by the exterior angle theorem, we can see that $\angle EDB = 22.5^\circ$. Thus $\triangle BDE$ is isosceles, as it has two equal angles, and so $DE = EB$. Thus $AB = AD + DE$, and since $\triangle ADE$ is a right isosceles triangle, it forms one half of a square with $AD$ the diagonal.

To complete construction of the square, we drop perpendiculars to $AC$ at $A$ and $ED$ at $E$. The point of intersection of these two perpendiculars gives us the location of the fourth vertex of the square, $F$.

5. We need to count how many subsets $T$ have $x_{min} = n$ for each integer $1 \le n \le 10$. For instance, $x_{min} = 10$ for only one subset, $\{10, 11, 12, 13, 14, 15, 16, 17\}$. To create an 8-element subset with $x_{min} = n$, imagine writing the elements of the subset in order. Clearly the first number is $n$, but we can choose the remaining 7 elements from $(17-n)$ options. Thus the number of subsets with $x_{min} = n$ is given by ${}^{17-n}C_7$.

So the arithmetic mean of the numbers selected is

$$\begin{aligned} x &= \frac{1}{24\,310}(1 \times {}^{16}C_7 + 2 \times {}^{15}C_7 + 3 \times {}^{14}C_7 + \cdots + 10 \times {}^{7}C_7) \\ &= \frac{1}{24\,310}(11\,440 + 12\,870 + 10\,296 + 6864 + 3960 + 1980 + 840 + 288 + 72 + 10) \\ &= \frac{48620}{24\,310} = 2 \end{aligned}$$

## Senior Questions

1. (a) Differentiating to find the stationary points we have

$$f'(x) = 3ax^2 + 2bx + c$$

And solving for $3ax^2 + 2bx + c = 0$,

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \end{aligned}$$

(b) If $b^2 - 3ac < 0$, then the cubic has no stationary points. If $b = 0$, the cubic will have one stationary point if $c = 0$, or two stationary points if $a$ and $c$ have opposite signs.

(c) Firstly, let's find the coordinates of the point of inflexion.

$$f''(x) = 6ax + 2b$$

So if $f''(x) = 0$, then $x = -\frac{b}{2a}$, which we can see is the average of the $x$ coordinates of the two stationary points.

This occurs because a cubic has rotational symmetry about the point of inflexion. If we re-write the equation of the cubic in terms of the variables $X = x + \frac{b}{2a}$ and $Y = \frac{bc}{3a} - \frac{3b^3}{27a^2}$ (that is, if we translate the coordinate system so that the point of inflexion is at the origin), then the new equation becomes $Y = \alpha X^3 + \beta X$, which we can easily see is an odd function.

2. We use the binomial theorem to expand $x^3$. Then

$$\begin{aligned} x^3 &= (\sqrt[3]{10} + \sqrt[3]{6})^3 \\ &= 10 + 3(\sqrt[3]{10})^2\sqrt[3]{6} + 3\sqrt[3]{10}(\sqrt[3]{6})^2 + 6 \\ &= 16 + 3(\sqrt[3]{10} + \sqrt[3]{6})(\sqrt[3]{10})(\sqrt[3]{6}) \\ &= 16 + 3x\sqrt[3]{60} \\ \Rightarrow x^3 - 3x\sqrt[3]{60} &= 16 \end{aligned}$$

Now we can consider the polynomial $p(x) = x^3 - 3x\sqrt[3]{60} - 16$. Then $x$ is a zero of $p$, and we must show that there are no zeros of $p$ that are larger than 4. To do this, we will show that $p(4) > 0$ and $p$ is monotonically increasing for $x \geq 4$. Firstly,

$$p(4) = 64 - 12\sqrt[3]{60} - 16 = 48 - 12\sqrt[3]{60}$$

Since $3^3 < 60 < 4^3$, $\sqrt[3]{60} < 4$ and so $p(4) > 0$. Furthermore, $p'(x) = 3x^2 - 3\sqrt[3]{60} = 3(x^2 - \sqrt[3]{60})$, and if $x > 4$, then $p'(x) > 0$. Thus $p(x)$ has no roots larger than 4. Hence $x < 4$.